Complete Solutions to Exercise 15b

1. (a) We are given the simple continued fraction $[1;\ 1,\ 1,\ 1,\ 1]$. Generating a table to find the values gives:

k	a_k	p_k	q_k
−1		0	1
0		1	0
1	1	$(1\times1)+0=1$	$(1\times0)+1=1$
2	1	$(1\times1)+1=2$	$(1\times1)+0=1$
3	1	$(1\times2)+1=3$	$(1\times1)+1=2$
4	1	$(3\times1)+2=5$	$(2\times1)+1=3$
5	1	$(5\times1)+3=8$	$(1\times3)+2=5$

The convergents are given by

$$C_1=\frac{p_1}{q_1}=\frac{1}{1}=1,\ C_2=\frac{p_2}{q_2}=\frac{2}{1}=2,\ C_3=\frac{p_3}{q_3}=\frac{3}{2},\ C_4=\frac{p_4}{q_4}=\frac{5}{3} \text{ and } C_5=\frac{p_5}{q_5}=\frac{8}{5}$$

In decimal format $C_5=\frac{8}{5}=1.60$ (2dp).

(b) We need to find the first 5 convergents of $[2;\ 2,\ 2,\ 2,\ 2]$. Similarly to part (a) we have

k	a_k	p_k	q_k
−1		0	1
0		1	0
1	2	$(2\times1)+0=2$	$(2\times0)+1=1$
2	2	$(2\times2)+1=5$	$(2\times1)+0=2$
3	2	$(2\times5)+2=12$	$(2\times2)+1=5$
4	2	$(2\times12)+5=29$	$(2\times5)+2=12$
5	2	$(2\times29)+12=70$	$(2\times12)+5=29$

The convergents are given by

$$C_1=\frac{p_1}{q_1}=\frac{2}{1}=2,\ C_2=\frac{p_2}{q_2}=\frac{5}{2},\ C_3=\frac{p_3}{q_3}=\frac{12}{5},\ C_4=\frac{p_4}{q_4}=\frac{29}{12} \text{ and } C_5=\frac{p_5}{q_5}=\frac{70}{29}$$

We have $C_5=\frac{70}{29}=2.41$ (2dp).

(c) This time we are interested in evaluating the convergents of $[1;\ 2,\ 3,\ 4,\ 5]$:

k	a_k	p_k	q_k
−1		0	1
0		1	0
1	1	$(1\times1)+0=1$	$(1\times0)+1=1$
2	2	$(2\times1)+1=3$	$(2\times1)+0=2$
3	3	$(3\times3)+1=10$	$(3\times2)+1=7$
4	4	$(4\times10)+3=43$	$(4\times7)+2=30$

5	5	$(5\times 43)+10=225$	$(5\times 30)+7=157$

The convergents are given by

$$C_1=\frac{p_1}{q_1}=\frac{1}{1}=1,\ C_2=\frac{p_2}{q_2}=\frac{3}{2},\ C_3=\frac{p_3}{q_3}=\frac{10}{7},\ C_4=\frac{p_4}{q_4}=\frac{43}{30} \text{ and } C_5=\frac{p_5}{q_5}=\frac{225}{157}$$

Hence $C_5=\frac{225}{157}=1.43$ (2dp).

(d) We are given $[6;\ 7,\ 8,\ 9,\ 10]$. Creating a table to find the convergents:

k	a_k	p_k	q_k
−1		0	1
0		1	0
1	6	$(6\times 1)+0=6$	$(6\times 0)+1=1$
2	7	$(7\times 6)+1=43$	$(7\times 1)+0=7$
3	8	$(8\times 43)+6=350$	$(8\times 7)+1=57$
4	9	$(9\times 350)+43=3193$	$(9\times 57)+7=520$
5	10	$(10\times 3193)+350=32280$	$(10\times 520)+57=5257$

The convergents are given by

$$C_1=\frac{p_1}{q_1}=\frac{6}{1}=6,\ C_2=\frac{p_2}{q_2}=\frac{43}{7},\ C_3=\frac{p_3}{q_3}=\frac{350}{57},\ C_4=\frac{p_4}{q_4}=\frac{3193}{520} \text{ and } C_5=\frac{p_5}{q_5}=\frac{32280}{5257}$$

The last convergent in decimal format is $C_5=\frac{32280}{5257}=6.14$ (2dp).

2. We are given $e=[2;\ 1,\ 2,\ 1,\ 1,\ 4,\ 1,\ 1,\ 6,\ 1,\ 1,\ \cdots]$ and we are required to find the first 9 convergents:

k	a_k	p_k	q_k
−1		0	1
0		1	0
1	2	$(2\times 1)+0=2$	$(2\times 0)+1=1$
2	1	$(1\times 2)+1=3$	$(1\times 1)+0=1$
3	2	$(2\times 3)+2=8$	$(2\times 1)+1=3$
4	1	$(1\times 8)+3=11$	$(1\times 3)+1=4$
5	1	$(1\times 11)+8=19$	$(1\times 4)+3=7$
6	4	$(4\times 19)+11=87$	$(4\times 7)+4=32$
7	1	$(1\times 87)+19=106$	$(1\times 32)+7=39$
8	1	$(1\times 106)+87=193$	$(1\times 39)+32=71$
9	6	$(6\times 193)+106=1264$	$(6\times 71)+39=465$

The first 9 convergents are:

$$C_1=\frac{p_1}{q_1}=\frac{2}{1}=2,\ C_2=\frac{p_2}{q_2}=\frac{3}{1}=3,\ C_3=\frac{p_3}{q_3}=\frac{8}{3},\ C_4=\frac{p_4}{q_4}=\frac{11}{4},\ C_5=\frac{p_5}{q_5}=\frac{19}{7},$$

$$C_6 = \frac{p_6}{q_6} = \frac{87}{32},\ C_7 = \frac{p_7}{q_7} = \frac{106}{39},\ C_8 = \frac{p_8}{q_8} = \frac{193}{71},\ C_9 = \frac{p_9}{q_9} = \frac{1264}{465}$$

The 9th convergent in decimal format is $C_9 = \frac{1264}{465} = 2.718278$ (6 dp).

3. We need to find rational approximations to $\text{¤} = \frac{1+\sqrt{5}}{2} = [1;\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ \cdots]$:

k	a_k	p_k	q_k
−1		0	1
0		1	0
1	1	$(1\times1)+0=1$	$(1\times0)+1=1$
2	1	$(1\times1)+1=2$	$(1\times1)+0=1$
3	1	$(1\times2)+1=3$	$(1\times1)+1=2$
4	1	$(1\times3)+2=5$	$(1\times2)+1=3$
5	1	$(1\times5)+3=8$	$(1\times3)+2=5$
6	1	$(1\times8)+5=13$	$(1\times5)+3=8$
7	1	$(1\times13)+8=21$	$(1\times8)+5=13$
8	1	$(1\times21)+13=34$	$(1\times13)+8=21$
9	1	$(1\times34)+21=55$	$(1\times21)+13=34$

The first 9 convergents are:

$$C_1 = \frac{p_1}{q_1} = \frac{1}{1} = 1,\ \ C_2 = \frac{p_2}{q_2} = \frac{2}{1} = 2,\ \ C_3 = \frac{p_3}{q_3} = \frac{3}{2},\ C_4 = \frac{p_4}{q_4} = \frac{5}{3},\ C_5 = \frac{p_5}{q_5} = \frac{8}{5},$$

$$C_6 = \frac{p_6}{q_6} = \frac{13}{8},\ C_7 = \frac{p_7}{q_7} = \frac{21}{13},\ C_8 = \frac{p_8}{q_8} = \frac{34}{21},\ C_9 = \frac{p_9}{q_9} = \frac{55}{34}$$

The 9th convergent in decimal format is $C_9 = \frac{55}{34} = 1.617647$ (6 dp).

4. (a) We are required to find the simple continued fraction of $\sqrt{5}$. Creating a table:

Step n	Equation for $\sim_n$	$\sim_n$ (Calculator display)	$a_n = \lfloor \sim_n \rfloor$
1	$\sim_1 = \sqrt{5}$	$2.236067977\cdots$	2
2	$\sim_2 = \frac{1}{\sim_1 - a_1} = \frac{1}{2.236067977\cdots - 2}$	$4.236067977\cdots$	4
3	$\sim_3 = \frac{1}{\sim_2 - a_2} = \frac{1}{4.236067977\cdots - 4}$	$4.236067977\cdots$	4
4	$\sim_4 = \frac{1}{\sim_3 - a_3} = \frac{1}{4.236067977\cdots - 4}$	$4.236067977\cdots$	4
5	$\sim_5 = \frac{1}{\sim_4 - a_4} = \frac{1}{4.236067977\cdots - 4}$	$4.236067977\cdots$	4

6	$\sim_6 = \dfrac{1}{\sim_5 - a_5} = \dfrac{1}{4.236067977\cdots - 4}$	$4.236067977\cdots$	4
⋮	⋮	⋮	⋮

The simple continued fraction expansion of $\sqrt{5}$ is $\sqrt{5} = \left[2;\ \langle 4 \rangle\right]$.

(b) We need to find the simple continued fraction of $\sqrt{3}$:

We have $\sqrt{3} = 1.732050808\cdots$. Taking the floor function of this gives

$$a_1 = \lfloor \sim_1 \rfloor = \lfloor 1.732050808\cdots \rfloor = 1$$

Substituting $n = 1$ into $\sim_{n+1} = \dfrac{1}{\sim_n - a_n}$ gives $\sim_{1+1} = \dfrac{1}{\sim_1 - a_1}$. Putting $a_1 = 1$ and $\sim_1 = 1.732050808\cdots$ into this yields:

$$\sim_2 = \frac{1}{\sim_1 - a_1} = \frac{1}{1.732050808\cdots - 1} = \frac{1}{0.732050808\cdots} = 1.366025404\cdots$$

Hence $a_2 = \lfloor \sim_2 \rfloor = \lfloor 1.366025404\cdots \rfloor = 1$.

Repeating this process we have

$$\sim_3 = \frac{1}{\sim_2 - a_2} = \frac{1}{1.366025404\cdots - 1} = \frac{1}{0.366025404\cdots} = 2.732050806\cdots$$

Therefore $a_3 = \lfloor \sim_3 \rfloor = \lfloor 2.732050806\cdots \rfloor = 2$.

Repeating this process we have

$$\sim_4 = \frac{1}{\sim_3 - a_3} = \frac{1}{2.732050806\cdots - 2} = \frac{1}{0.732050806\cdots} = 1.366025404\cdots$$

Note that $\sim_4 = \sim_2 = 1.366025404\cdots$. This implies that $a_4 = a_2 = 1$.

From this we have $\sim_3 = \sim_5 = 2.732050806\cdots$ and so $a_5 = a_3 = 2$.

This means that

$$a_3 = a_5 = a_7 = \cdots = 1$$
$$a_2 = a_4 = a_6 = a_8 = \cdots = 2$$

Hence the continued fraction expansion of $\sqrt{3}$ is $\left[1; \langle 1,\ 2 \rangle\right]$.

(c) We find the simple continued fraction of $\sqrt{7}$ by creating a table:

Step n	Equation for $\sim_n$	$\sim_n$ (Calculator display)	$a_n = \lfloor \sim_n \rfloor$
1	$\sim_1 = \sqrt{7}$	$2.645751311\cdots$	2
2	$\sim_2 = \dfrac{1}{\sim_1 - a_1} = \dfrac{1}{2.645751311\cdots - 2}$	$1.54858377\cdots$	1
3	$\sim_3 = \dfrac{1}{\sim_2 - a_2} = \dfrac{1}{1.54858377\cdots - 1}$	$1.822875656\cdots$	1
4	$\sim_4 = \dfrac{1}{\sim_3 - a_3} = \dfrac{1}{1.822875656\cdots - 1}$	$1.215250437\cdots$	1
5	$\sim_5 = \dfrac{1}{\sim_4 - a_4} = \dfrac{1}{1.215250437\cdots - 1}$	$4.645751311\cdots$	4

6	$\sim_6 = \frac{1}{\sim_5 - a_5} = \frac{1}{4.645751311\cdots - 4}$	$1.54858377\cdots$	4
7	$\sim_7 = \frac{1}{\sim_6 - a_6} = \frac{1}{1.54858377\cdots - 1}$	$1.822875656\cdots$	1
8	$\sim_8 = \frac{1}{\sim_7 - a_7} = \frac{1}{1.822875656\cdots - 1}$	$1.215250437\cdots$	1
⋮	⋮	⋮	⋮

Note the shaded entries are going to repeat therefore

$$\sqrt{7} = \left[2; \langle 1,\ 1,\ 1,\ 4 \rangle\right]$$

5. (a) We need to find the simple continued fraction of $\frac{1+\sqrt{3}}{2}$:

Step n	Equation for $\sim_n$	$\sim_n$ (Calculator display)	$a_n = \lfloor \sim_n \rfloor$
1	$\sim_1 = \frac{1+\sqrt{3}}{2}$	$1.366025404\cdots$	1
2	$\sim_2 = \frac{1}{\sim_1 - a_1} = \frac{1}{1.366025404\cdots - 1}$	$2.732050808\cdots$	2
3	$\sim_3 = \frac{1}{\sim_2 - a_2} = \frac{1}{2.732050808\cdots - 2}$	$1.366025404\cdots$	1
4	$\sim_4 = \frac{1}{\sim_3 - a_3} = \frac{1}{1.366025404\cdots - 1}$	$2.732050808\cdots$	2
⋮	⋮	⋮	⋮

Note that this is going to repeat, therefore

$$\frac{1+\sqrt{3}}{2} = \left[\langle 1,\ 2 \rangle\right]$$

We have to find the first 5 convergents of $\left[\langle 1,\ 2 \rangle\right]$.

k	a_k	p_k	q_k
−1		0	1
0		1	0
1	1	$(1\times 1)+0=1$	$(1\times 0)+1=1$
2	2	$(2\times 1)+1=3$	$(2\times 1)+0=2$
3	1	$(1\times 3)+1=4$	$(1\times 2)+1=3$
4	2	$(2\times 4)+3=11$	$(2\times 3)+2=8$
5	1	$(1\times 11)+4=15$	$(1\times 8)+3=11$

The 5 convergents are

$$C_1 = \frac{p_1}{q_1} = \frac{1}{1} = 1,\quad C_2 = \frac{p_2}{q_2} = \frac{3}{2},\quad C_3 = \frac{p_3}{q_3} = \frac{4}{3},\quad C_4 = \frac{p_4}{q_4} = \frac{11}{8} \text{ and } C_5 = \frac{p_5}{q_5} = \frac{15}{11}$$

(b) Similarly for $\frac{2\sqrt{3}+3}{3}$ we have

Step n	Equation for $\sim_n$	$\sim_n$ (Calculator display)	$a_n = \lfloor \sim_n \rfloor$
1	$\sim_1 = \dfrac{2\sqrt{3}+3}{3}$	$2.154700538\cdots$	2
2	$\sim_2 = \dfrac{1}{\sim_1 - a_1} = \dfrac{1}{2.154700538\cdots - 2}$	$6.464101615\cdots$	6
3	$\sim_3 = \dfrac{1}{\sim_2 - a_2} = \dfrac{1}{6.464101615\cdots - 6}$	$2.154700538\cdots$	2
4	$\sim_4 = \dfrac{1}{\sim_3 - a_3} = \dfrac{1}{2.154700538\cdots - 2}$	$6.464101615\cdots$	6
$\vdots$	$\vdots$	$\vdots$	$\vdots$

The $\sim_n$ of step 2 and 4 are identical so the terms are going to repeat because they depend on the previous terms. Hence the continued fraction is

$$\frac{2\sqrt{3}+3}{3} = \left[2;\ \langle 6,\ 2\rangle\right]$$

We have to find the first 5 convergents of $\left[2;\ \langle 6,\ 2\rangle\right]$.

$\boldsymbol{k}$	a_k	p_k	q_k
−1		0	1
0		1	0
1	2	$(2\times1)+0=2$	$(2\times0)+1=1$
2	6	$(6\times2)+1=13$	$(6\times1)+0=6$
3	2	$(2\times13)+2=28$	$(2\times6)+1=13$
4	6	$(6\times28)+13=181$	$(6\times13)+6=84$
5	2	$(2\times181)+28=390$	$(2\times84)+13=181$

The 5 convergents are

$$C_1 = \frac{p_1}{q_1} = \frac{2}{1} = 2,\quad C_2 = \frac{p_2}{q_2} = \frac{13}{6},\quad C_3 = \frac{p_3}{q_3} = \frac{28}{13},\quad C_4 = \frac{p_4}{q_4} = \frac{181}{84} \text{ and } C_5 = \frac{p_5}{q_5} = \frac{390}{181}$$

(c) Creating a table of values for $\dfrac{2+\sqrt{7}}{3}$:

Step n	Equation for $\sim_n$	$\sim_n$ (Calculator display)	$a_n = \lfloor \sim_n \rfloor$
1	$\sim_1 = \dfrac{2+\sqrt{7}}{3}$	$1.54858377\cdots$	1
2	$\sim_2 = \dfrac{1}{\sim_1 - a_1} = \dfrac{1}{1.54858377\cdots - 1}$	$1.822875656\cdots$	1
3	$\sim_3 = \dfrac{1}{\sim_2 - a_2} = \dfrac{1}{1.822875656\cdots - 1}$	$1.215250437\cdots$	1

4	$\sim_4 = \frac{1}{\sim_3 - a_3} = \frac{1}{1.215250437\cdots - 1}$	$4.645751311\cdots$	4
5	$\sim_5 = \frac{1}{\sim_4 - a_4} = \frac{1}{4.645751311\cdots - 4}$	$1.54858377\cdots$	1

Note that at steps 1 and 5 the a_n 's are equal. Since the terms are derived from the previous terms so we are going to see the pattern repeated. Hence

$$\frac{2+\sqrt{7}}{3} = \left[\langle 1,\ 1,\ 1,\ 4\rangle\right]$$

We use the following table to find the convergents:

k	a_k	p_k	q_k
−1		0	1
0		1	0
1	1	$(1\times1)+0=1$	$(1\times0)+1=1$
2	1	$(1\times1)+1=2$	$(1\times1)+0=1$
3	1	$(1\times2)+1=3$	$(1\times1)+1=2$
4	4	$(4\times3)+2=14$	$(4\times2)+1=9$
5	1	$(1\times14)+3=17$	$(1\times9)+2=11$

The 5 convergents are

$$C_1 = \frac{p_1}{q_1} = \frac{1}{1} = 1,\quad C_2 = \frac{p_2}{q_2} = \frac{2}{1} = 2,\quad C_3 = \frac{p_3}{q_3} = \frac{3}{2},\quad C_4 = \frac{p_4}{q_4} = \frac{14}{9} \text{ and } C_5 = \frac{p_5}{q_5} = \frac{17}{11}$$

6. (a) We need to find the irrational number $\sim = \left[\langle 1,\ 2\rangle\right]$. Writing the continued fraction out:

$$\sim = 1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \ddots}}}}}}$$

The part of the expression which is blue is exactly the original continued fraction. Hence the blue expression is equal to $\sim$. The above becomes

$$\sim = 1 + \cfrac{1}{2 + \cfrac{1}{\sim}}$$

Multiplying the last term on the Right Hand Side by $\frac{\sim}{\sim}(=1)$ gives

$$\sim = 1 + \frac{\sim}{2\sim + 1}$$

Multiplying both sides by $2\sim + 1$ yields

$$\sim(2\sim+1)=(2\sim+1)+\sim$$
$$2\sim^2+\sim=3\sim+1 \qquad [\text{Expanding}]$$
$$2\sim^2-2\sim-1=0$$

Using the quadratic formula to solve $2\sim^2-2\sim-1=0$ with $a=2,\ b=-2$ and $c=-1$:

$$\sim=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{2\pm\sqrt{(-2)^2-[4\times2\times(-1)]}}{2\times2}$$
$$=\frac{2\pm\sqrt{12}}{4}=\frac{2\pm2\sqrt{3}}{4}=\frac{1}{2}\left(1\pm\sqrt{3}\right)$$

$\sim$ *cannot* be negative so our only solution is

$$\sim=\frac{1}{2}\left(1+\sqrt{3}\right)$$

(b) We are asked to find the irrational number given by the continued fraction $\sim=[\langle 2,\ 1\rangle]$. Writing out the continued fraction:

$$\sim=2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{\ddots}}}}}}}$$

The part of the expression which is blue is exactly the original continued fraction. Hence the blue expression is equal to $\sim$. The above becomes

$$\sim=2+\cfrac{1}{1+\cfrac{1}{\sim}}$$

Multiplying the last term on the right hand side by $\frac{\sim}{\sim}(=1)$ gives

$$\sim=2+\frac{\sim}{\sim+1}$$

Multiplying both sides by $\sim+1$ yields

$$\sim(\sim+1)=2(\sim+1)+\sim$$
$$\sim^2+\sim=2\sim+2+\sim$$
$$\sim^2-2\sim-2=0$$

Using the quadratic formula to solve $\sim^2-2\sim-2=0$ with $a=1,\ b=-2$ and $c=-2$:

$$\sim=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{2\pm\sqrt{(-2)^2-[4\times1\times(-2)]}}{2\times1}$$
$$=\frac{2\pm\sqrt{12}}{2}=\frac{2\pm\sqrt{4\times3}}{2}=\frac{2\pm2\sqrt{3}}{2}=1\pm\sqrt{3}$$

$\sim$ *cannot* be negative so our only solution is $\sim=1+\sqrt{3}$.

(c) We are given $\sim=[2,\ \langle 5,\ 1\rangle]$. Writing out the continued fraction:

$$\sim\ = 2+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{1+\ddots}}}}}}$$

We can write the repeated continued fraction as another symbol like . This means we have

$$\sim\ = 2+\frac{1}{\ } \qquad\qquad (*)$$

If we can find   then we can determine $\sim$. *How can we find ?*

Since this is a repeated continued fraction:

$$= 5+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{5+\ddots}}}}}}$$

Again the continued fraction is copied out again as shown in blue. We have

$$= 5+\cfrac{1}{1+\cfrac{1}{\ }}$$

Repeating the same algebraic technique as in the above example we have

$$= 5+\frac{\ }{\ +1} \qquad \left[\text{Multiplying the last term by } \frac{\ }{\ } = 1\right]$$

$$(\ +1) = 5(\ +1)+ \qquad \left[\text{Multiplying both sides by } \ +1\right]$$

$$^2 + \ = 5\ + 5 +$$

$$^2 - 5\ - 5 = 0$$

Using the quadratic formula with $a=1,\ b=-5$ and $c=-5$ gives

$$= \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{5\pm\sqrt{(-5)^2-[4\times1\times(-5)]}}{2}$$

$$= \frac{5\pm\sqrt{45}}{2} = \frac{5\pm\sqrt{9\times5}}{2} = \frac{5\pm3\sqrt{5}}{2}$$

We ignore the negative solution so we have $= \frac{5+3\sqrt{5}}{2}$. Substituting this (*) yields

$$\sim\ = 2+\frac{1}{\ } = 2+\cfrac{1}{\cfrac{5+3\sqrt{5}}{2}} = 2+\frac{2}{5+3\sqrt{5}}$$

Rationalizing the denominator of this by multiplying the last term on the right hand side by $\frac{5-3\sqrt{5}}{5-3\sqrt{5}}(=1)$:

$$\sim \; = 2+\frac{2}{\left(5+3\sqrt{5}\right)}\frac{\left(5-3\sqrt{5}\right)}{\left(5-3\sqrt{5}\right)} = 2+\frac{10-6\sqrt{5}}{25-9\times 5}$$

$$= 2+\frac{10-6\sqrt{5}}{-20}$$

$$= 2+\frac{6\sqrt{5}-10}{20}$$

$$= \frac{2(20)+6\sqrt{5}-10}{20} = \frac{30+6\sqrt{5}}{20} = \frac{15+3\sqrt{5}}{10}$$