## Complete Solutions to Exercise 15c

1. (a) We are given the simple continued fraction $[1;\ \langle 2\rangle]$. Generating a table to find the convergent values gives:

| **k** | $a_k$ | $p_k$ | $q_k$ |
|---|---|---|---|
| **−1** | | 0 | 1 |
| **0** | | 1 | 0 |
| **1** | 1 | $(1\times1)+0=1$ | $(1\times0)+1=1$ |
| **2** | 2 | $(2\times1)+1=3$ | $(2\times1)+0=2$ |
| **3** | 2 | $(3\times2)+1=7$ | $(2\times2)+1=5$ |
| **4** | 2 | $(2\times7)+3=17$ | $(2\times5)+2=12$ |
| **5** | 2 | $(2\times17)+7=41$ | $(2\times12)+5=29$ |

The convergents are given by

$C_1=\frac{p_1}{q_1}=\frac{1}{1}=1$, $C_2=\frac{p_2}{q_2}=\frac{3}{2}$, $C_3=\frac{p_3}{q_3}=\frac{7}{5}$, $C_4=\frac{p_4}{q_4}=\frac{17}{12}$ and $C_5=\frac{p_5}{q_5}=\frac{41}{29}$

The difference is given by

$$\left|\sqrt{2}-\frac{41}{29}\right|=4.2\times10^{-4}\ \text{(2sf)}$$

(b) This time we are given $\sqrt{5}=[2;\ \langle 4\rangle]$. The table for this is

| **k** | $a_k$ | $p_k$ | $q_k$ |
|---|---|---|---|
| **−1** | | 0 | 1 |
| **0** | | 1 | 0 |
| **1** | 2 | $(2\times1)+0=2$ | $(2\times0)+1=1$ |
| **2** | 4 | $(4\times2)+1=9$ | $(4\times1)+0=4$ |
| **3** | 4 | $(4\times9)+2=38$ | $(4\times4)+1=17$ |
| **4** | 4 | $(4\times38)+9=161$ | $(4\times17)+4=72$ |
| **5** | 4 | $(4\times161)+38=682$ | $(4\times72)+17=305$ |

The convergents are given by

$C_1=\frac{p_1}{q_1}=\frac{2}{1}=2$, $C_2=\frac{p_2}{q_2}=\frac{9}{4}$, $C_3=\frac{p_3}{q_3}=\frac{38}{17}$, $C_4=\frac{p_4}{q_4}=\frac{161}{72}$, $C_5=\frac{p_5}{q_5}=\frac{682}{305}$

The difference is given by

$$\left|\sqrt{5}-\frac{682}{305}\right|=2.4\times10^{-6}$$

(c) We need to find the convergents of $\sqrt{3}=[1;\ \langle 1,\ 2\rangle]$:

| $k$ | $a_k$ | $p_k$ | $q_k$ |
|---|---|---|---|
| **−1** | | 0 | 1 |
| **0** | | 1 | 0 |
| **1** | 1 | $(1\times1)+0=1$ | $(1\times0)+1=1$ |
| **2** | 1 | $(1\times1)+1=2$ | $(1\times1)+0=1$ |
| **3** | 2 | $(2\times2)+1=5$ | $(2\times1)+1=3$ |
| **4** | 1 | $(1\times5)+2=7$ | $(1\times3)+1=4$ |
| **5** | 2 | $(2\times7)+5=19$ | $(2\times4)+3=11$ |

The convergents are given by

$$C_1=\frac{p_1}{q_1}=\frac{1}{1}=1,\ C_2=\frac{p_2}{q_2}=\frac{2}{1}=2,\ C_3=\frac{p_3}{q_3}=\frac{5}{3},\ C_4=\frac{p_4}{q_4}=\frac{7}{4},\ C_5=\frac{p_5}{q_5}=\frac{19}{11}$$

The difference is given by

$$\left|\sqrt{3}-\frac{19}{11}\right|=4.8\times10^{-3}$$

This error is nearly 5 parts in a thousand.

The largest error is $\sqrt{3}$ with $\frac{19}{11}$.

2. (i) We use the Euclidean algorithm to find the continued fraction of $\frac{43}{5}$:

$$43=8(5)+3$$
$$5=1(3)+2$$
$$3=1(2)+1$$
$$2=2(1)$$

The simple continued fraction is given by the quotients,

$$\frac{43}{5}=[8;\ 1,\ 1,\ 2]$$

(ii) The convergents $C_1$, $C_2$, $C_3$ and $C_4$ are given by:

| **Step $n$** | $a_n$ | $p_n$ | $q_n$ |
|---|---|---|---|
| **−1** | | 0 | 1 |
| **0** | | 1 | 0 |
| **1** | 8 | $(8\times1)+0=8$ | $(8\times0)+1=1$ |
| **2** | 1 | $(1\times8)+1=9$ | $(1\times1)+0=1$ |
| **3** | 1 | $(1\times9)+8=17$ | $(1\times1)+1=2$ |
| **4** | 2 | $(2\times17)+9=43$ | $(2\times2)+1=5$ |

The determinant of the highlighted entries is

$$\det\begin{pmatrix}17 & 2\\ 43 & 5\end{pmatrix}=5(17)-43(2)=(-1)^{4+1}=-1$$

Multiplying through by $-1$ gives

$$5(-17)+43(2)=1 \qquad (\dagger)$$

We were asked to solve $43x+5y=7$. Multiplying this (†) by 7 yields

$$5(-17\times 7)+43(2\times 7)=5(-119)+43(14)=7$$

Hence a particular solution is $x_0=14,\ y_0=-119$.

The general solution is given by Corollary (1.18) of chapter 1:

> Let $\gcd(a,\ b)=1$ and $x_0$, $y_0$ be particular solutions of the equation
>
> $$ax+by=c$$
>
> Then *all* the other solutions of this equation are given by
>
> $$x=x_0+bt \text{ and } y=y_0-at$$

Using this with $a=43,\ b=5,\ x_0=14,\ y_0=-119$ we have

$$x=14+5t \text{ and } y=-119-43t$$

3. (i) We use the Euclidean algorithm to find the simple continued fraction of $\frac{106}{19}$:

$$106=5(19)+11$$
$$19=1(11)+8$$
$$11=1(8)+3$$
$$8=2(3)+2$$
$$3=1(2)+1$$
$$2=2(1)$$

The continued fraction of $\frac{106}{19}$ is $[5;\ 1,\ 1,\ 2,\ 1,\ 2]$.

(ii) The convergents are given by:

| **Step *n*** | $a_n$ | $p_n$ | $q_n$ |
|---|---|---|---|
| **−1** | | 0 | 1 |
| **0** | | 1 | 0 |
| **1** | 5 | $(5\times 1)+0=5$ | $(5\times 0)+1=1$ |
| **2** | 1 | $(1\times 5)+1=6$ | $(1\times 1)+0=1$ |
| **3** | 1 | $(1\times 6)+5=11$ | $(1\times 1)+1=2$ |
| **4** | 2 | $(2\times 11)+6=28$ | $(2\times 2)+1=5$ |
| **5** | 1 | $(1\times 28)+11=39$ | $(1\times 5)+2=7$ |
| **6** | 2 | $(2\times 39)+28=106$ | $(2\times 7)+5=19$ |

The determinant of the highlighted entries is given by

$$\det\begin{pmatrix}39 & 7\\ 106 & 19\end{pmatrix}=19(39)-106(7)=(-1)^{6+1}=-1$$

Multiplying this by $-1$ yields

$$19(-39)+106(7)=1 \qquad (*)$$

We need to solve $106x+19y=100$. Multiplying (*) by 100 gives

$$19(-3900)+106(700)=100$$

A particular solution is $x_0=700,\ y_0=-3900$. We can find the general solution by

Let $\gcd(a,\ b)=1$ and $x_0$, $y_0$ be particular solutions of the equation

$$ax+by=c$$

Then *all* the other solutions of this equation are given by

$$x=x_0+bt \text{ and } y=y_0-at$$

Using this with $a=106,\ b=19,\ x_0=700,\ y_0=-3900$ we have

$$x=700+19t \text{ and } y=-3900-106t$$

4. The solutions are very similar to question 2 and 3.

(a) We are given $86x+17y=3$. By using the Euclidean algorithm we find the continued fraction of $\frac{86}{17}$:

$$86=5(17)+1$$

$$17=17(1)$$

The continued fraction is $[5;\ 17]$. We have

$$\frac{86}{17}=5+\frac{1}{17}$$

The convergents are $C_1=\frac{5}{1}=\frac{p_1}{q_1}$ and $C_2=\frac{86}{17}=\frac{p_2}{q_2}$. From this we have

$$p_1q_2-p_2q_1=5(17)-86(1)=-1$$

Multiplying this by $-1$ yields $86(1)-17(5)=1$.

Multiplying this by 3:

$$86(1\times3)-17(5\times3)=86(3)-17(15)=3$$

Hence a particular solution to $86x+17y=3$ is $x_0=3,\ y_0=-15$. For the general solution we use Corollary (1.18):

Let $\gcd(a,\ b)=1$ and $x_0$, $y_0$ be particular solutions of the equation

$$ax+by=c$$

Then *all* the other solutions of this equation are given by

$$x=x_0+bt \text{ and } y=y_0-at$$

Using this with $a=86,\ b=17, x_0=3,\ y_0=-15$ we have

$$x=x_0+bt=3+17t \text{ and } y=y_0-at=-15-86t$$

(b) Similarly to solve $111x+23y=5$ we find the continued fraction of $\frac{111}{23}$:

$$111=4(23)+19$$

$$23=1(19)+4$$

$$19=4(4)+3$$

$$4=1(3)+1$$

$$3=3(1)$$

The continued fraction of $\frac{111}{23}$ is $[4;\ 1,\ 4,\ 1,\ 3]$.

The convergents are given by:

| **Step *n*** | $a_n$ | $p_n$ | $q_n$ |
|---|---|---|---|
| **−1** | | 0 | 1 |
| **0** | | 1 | 0 |
| **1** | 4 | $(4\times1)+0=4$ | $(4\times0)+1=1$ |
| **2** | 1 | $(1\times4)+1=5$ | $(1\times1)+0=1$ |
| **3** | 4 | $(4\times5)+4=24$ | $(4\times1)+1=5$ |
| **4** | 1 | $(1\times24)+5=29$ | $(1\times5)+1=6$ |
| **5** | 3 | $(3\times29)+24=111$ | $(3\times6)+5=23$ |

The determinant is given by

$$\det\begin{pmatrix}29 & 6\\ 111 & 23\end{pmatrix}=23(29)-111(6)=(-1)^{5+1}=1$$

We are required to solve $111x+23y=5$. Multiplying $23(29)-111(6)=1$ by 5 gives

$$23(29\times5)-111(6\times5)=23(145)-111(30)=5$$

Hence a particular solution is $x_0=-30,\ y_0=145$. For the general solution we use

$$x=x_0+bt \text{ and } y=y_0-at$$

With $a=111,\ b=23,\ x_0=-30,\ y_0=145$:

$$x=x_0+bt=-30+23t \text{ and } y=145-111t$$

(c) We need to find the continued fraction which helps in solving $201x+51y=9$. Since the $\gcd(201,\ 51)=3$ so we can solve the equivalent simpler equation

$$67x+17y=3$$

We need to find the continued fraction which represents $\frac{67}{17}$:

$$67=3(17)+16$$
$$17=1(16)+1$$
$$16=16(1)$$

The continued fraction of $\frac{67}{17}$ is $[3;\ 1,\ 16]$. Finding convergents gives

| **Step *n*** | $a_n$ | $p_n$ | $q_n$ |
|---|---|---|---|
| **−1** | | 0 | 1 |
| **0** | | 1 | 0 |
| **1** | 3 | $(3\times1)+0=3$ | $(3\times0)+1=1$ |
| **2** | 1 | $(1\times3)+1=4$ | $(1\times1)+0=1$ |
| **3** | 16 | $(16\times4)+3=67$ | $(16\times1)+1=17$ |

The determinant is

$$\det\begin{pmatrix}4 & 1\\ 67 & 17\end{pmatrix}=17(4)-67(1)=(-1)^{3+1}=1$$

We have $67(-1)+17(4)=1$. Since we need to solve $67x+17y=3$ so we multiply by 3:

$$67(-3)+17(12)=3$$

Hence a particular solution is $x_0=-3,\ y_0=12$. The general solution is found using

Let $\gcd(a,\ b)=1$ and $x_0$, $y_0$ be particular solutions of the equation

$$ax+by=c$$

Then *all* the other solutions of this equation are given by

$$x=x_0+bt \text{ and } y=y_0-at$$

Applying this with $a=67,\ b=17,\ x_0=-3,\ y_0=12$ gives

$$x=x_0+bt=-3+17t \text{ and } y=y_0-at=12-67t$$

5. We need to find the various convergents of the continued fraction and then use

Convergent Approximation Theorem (15.11).

$$\left|\sim-\frac{p_n}{q_n}\right|<\frac{1}{q_nq_{n+1}}$$

(a) We are given $\sqrt{10}=[3;\ \langle 6\rangle]$. The first few convergents are

| **k** | $a_k$ | $p_k$ | $q_k$ |
|---|---|---|---|
| **−1** | | 0 | 1 |
| **0** | | 1 | 0 |
| **1** | 3 | $(3\times1)+0=3$ | $(3\times0)+1=1$ |
| **2** | 6 | $(6\times3)+1=19$ | $(6\times1)+0=6$ |
| **3** | 6 | $(6\times19)+3=117$ | $(6\times6)+1=37$ |
| **4** | 6 | $(6\times117)+19=721$ | $(6\times37)+6=228$ |
| **5** | 6 | $(6\times721)+117=4443$ | $(6\times228)+37=1405$ |

We need to find the first convergent where

$$\left|\sqrt{10}-\frac{p_n}{q_n}\right|<0.0001$$

Well we have by the above theorem (15.11):

$$\left|\sqrt{10}-\frac{p_n}{q_n}\right|<\frac{1}{q_nq_{n+1}}$$

If we try $n=3$ then

$$\left|\sqrt{10}-\frac{117}{37}\right|<\frac{1}{37\times228}=0.000119 \text{ (3 sf)}$$

This is slightly over our limit of $0.0001$. It should be within our limit for the next $n=4$:

$$\left|\sqrt{10}-\frac{721}{228}\right|<\frac{1}{228\times1405}=3.12\times10^{-6}$$

The first convergent to be within our stated limit is $\frac{721}{228}$.

(b) This time we need to find the convergents associated with $\sqrt{15}=\left[3;\ \langle 1,\ 6\rangle\right]$:

| $k$ | $a_k$ | $p_k$ | $q_k$ |
|---|---|---|---|
| **−1** | | 0 | 1 |
| **0** | | 1 | 0 |
| **1** | 3 | $(3\times 1)+0=3$ | $(3\times 0)+1=1$ |
| **2** | 1 | $(1\times 3)+1=4$ | $(1\times 1)+0=1$ |
| **3** | 6 | $(6\times 4)+3=27$ | $(6\times 1)+1=7$ |
| **4** | 1 | $(1\times 27)+4=31$ | $(1\times 7)+1=8$ |
| **5** | 6 | $(6\times 31)+27=213$ | $(6\times 8)+7=55$ |
| **6** | 1 | $(1\times 213)+31=244$ | $(1\times 55)+8=63$ |
| **7** | 6 | $(6\times 244)+213=1677$ | $(6\times 63)+55=433$ |

Evaluating the last few $q$ values:

$$\frac{1}{q_5q_6}=\frac{1}{55\times 63}=0.00029>0.0001$$

$$\frac{1}{q_6q_7}=\frac{1}{63\times 433}=3.66\times 10^{-5}<0.0001$$

Hence the first convergent which is within 0.0001 of $\sqrt{15}$ is when $n=6$:

$$\left|\sqrt{15}-\frac{244}{63}\right|<\frac{1}{q_6q_7}=0.00013.66\times 10^{-5}<0.0001$$

(c) We need to find the first convergent where $\frac{1+\sqrt{10}}{3}=\left[1;\ \langle 2,\ 1,\ 1\rangle\right]$ is within 0.0001 of $\frac{1+\sqrt{10}}{3}$. This time we add another column $\frac{1}{q_kq_{k+1}}$. There is no point evaluating the first few of $\frac{1}{q_kq_{k+1}}$ because they will always be greater than 0.0001 so we have placed a * symbol.

| $k$ | $a_k$ | $p_k$ | $q_k$ | $\frac{1}{q_kq_{k+1}}$ |
|---|---|---|---|---|
| **−1** | | 0 | 1 | |
| **0** | | 1 | 0 | |
| **1** | 1 | $(1\times 1)+0=1$ | $(1\times 0)+1=1$ | * |
| **2** | 2 | $(2\times 1)+1=3$ | $(2\times 1)+0=2$ | * |
| **3** | 1 | $(1\times 3)+1=4$ | $(1\times 2)+1=3$ | * |
| **4** | 1 | $(1\times 4)+3=7$ | $(1\times 3)+2=5$ | * |

| | | | | |
|---|---|---|---|---|
| **5** | 2 | $(2\times7)+4=18$ | $(2\times5)+3=13$ | * |
| **6** | 1 | $(1\times18)+7=25$ | $(1\times13)+5=18$ | * |
| **7** | 1 | $(1\times25)+18=43$ | $(1\times18)+13=31$ | 0.0004 |
| **8** | 2 | $(2\times43)+25=111$ | $(2\times31)+18=80$ | 0.00011 |
| **9** | 1 | $(1\times111)+43=154$ | $(1\times80)+31=111$ | $4.72\times10^{-5}$ |
| **10** | 1 | $(1\times154)+111=265$ | $(1\times111)+80=191$ | * |

The first value in the last column to be less than 0.0001 is highlighted:

$$\left|\frac{1+\sqrt{10}}{3}-\frac{154}{111}\right|<4.72\times10^{-5}<0.0001$$

The first convergent is $\frac{154}{111}$.

6. We need to prove the following result:
If $q_k$ is the denominator of the $k$th convergent of $[a_0;\ a_1,\ a_2,\ \cdots,\ a_n]$ then

$$q_{k-1}\le q_k \text{ for } 1\le k\le n$$

*How do we prove this?*
By using mathematical induction.
*Proof.*
For $k=1$ we have $q_0$ where $q_0=0$. By (15.5) we have:

$$q_k=a_kq_{k-1}+q_{k-2}$$

Putting in $k=1$ yields $q_1=a_1q_0+q_{-1}=0+1=1$. Hence we have $q_0\le q_1$.
Assume the result is true for $k=m$, that is

$$q_{m-1}\le q_m$$

We are required to prove that $q_m\le q_{m+1}$. By (15.4) and (15.5) we have the table:

| **Step *n*** | $a_n$ | $p_n$ | $q_n$ | |
|---|---|---|---|---|
| ⋮ | | ⋮ | ⋮ | |
| $m-1$ | $a_{m-1}$ | $p_{m-1}$ | $q_{m-1}$ | |
| $\boldsymbol{m}$ | $a_m$ | $p_m$ | $q_m$ | |
| $m+1$ | $a_{m+1}$ | $a_{m+1}p_m+p_{m-1}$ | $a_{m+1}q_m+q_{m-1}$ ← | $q_{m+1}$ |
| ⋮ | ⋮ | ⋮ | ⋮ | |

Therefore

$$q_{m+1}=a_{m+1}q_m+q_{m-1}\ge a_{m+1}q_m\ge(1)q_m \quad [\text{because } a_{m+1}\ge1]$$

Hence by mathematical induction we have

$$q_{k-1}\le q_k \text{ for } 1\le k\le n$$

■

7. We need to prove the following $\left|\sim - \frac{p_n}{q_n}\right| < \frac{1}{q_n^{\;2}}$.

*Proof*.

Let $\sim$ be an irrational number and $\frac{p_n}{q_n}$ be the $n$th convergent to $\sim$.

By the Convergent Approximation Theorem (15.11):

$$\left|\sim - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}$$

Using the result of the previous question $q_{k-1} \le q_k$ for $1 \le k \le n$ we have

$$q_{n+1} \ge q_n$$

Substituting this inequality into the above

$$\left|\sim - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}} \le \frac{1}{q_n q_n} = \frac{1}{q_n^{\;2}}$$

This is our required result.

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8. We need to prove that an infinite continued fraction is an irrational number.

*Proof*.

This is proof given in Burton on page 323.

Let $\sim = [a_0;\ a_1,\ a_2,\ \cdots]$ be an infinite continued fraction. This means that

$$\sim = \lim_{n\to\infty}(C_n) = \lim_{n\to\infty}\left(\frac{p_n}{q_n}\right)$$

We know that $\sim$ lies strictly between $C_n$ and $C_{n+1}$ so we have

$$0 < |\sim - C_n| < |C_{n+1} - C_n| = \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right|$$

$$= \left|\frac{p_{n+1}q_n - p_n q_{n+1}}{q_{n+1}q_n}\right|$$

By (15.8):

$$p_{n-1}q_n - p_n q_{n-1} = (-1)^{n+1}$$

Applying this to the numerator $p_{n+1}q_n - p_n q_{n+1}$ in the above derivation:

$$0 < |\sim - C_n| < \left|\frac{p_{n+1}q_n - p_n q_{n+1}}{q_{n+1}q_n}\right| = \left|\frac{(-1)^{n+2}}{q_{n+1}q_n}\right| = \frac{1}{q_{n+1}q_n}$$

Suppose $\sim$ is a rational number. Then $\sim = \frac{a}{b}$ where $a,\ b$ are positive integers.

Therefore substituting this into the above gives

$$0 < |\sim - C_n| = \left|\frac{a}{b} - \frac{p_n}{q_n}\right| < \frac{1}{q_{n+1}q_n}$$

Multiplying this by $bq_n$ yields

$$0 < \left| \frac{a\cancel{b}q_n}{\cancel{b}} - \frac{p_n b\cancel{q_n}}{\cancel{q_n}} \right| = \left| aq_n - p_n b \right| < \frac{b\cancel{q_n}}{q_{n+1}\cancel{q_n}} = \frac{b}{q_{n+1}} \qquad (*)$$

By question 6 $q_n$ increases with *n* and is unbounded. We choose *n* so that

$$b < q_{n+1}$$

Substituting this inequality into (*) gives

$$0 < \left| aq_n - p_n b \right| < \frac{b}{q_{n+1}} < \frac{q_{n+1}}{q_{n+1}} = 1$$

We have $0 < \left| aq_n - p_n b \right| < 1$. This implies that there exists a positive integer between 0 and 1 which is impossible. Therefore our supposition ~ is a rational number must be wrong so ~ is irrational.

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9. We need to prove the following:
If the infinite continued fractions $[a_0;\ a_1,\ a_2,\ \cdots]$ and $[b_0;\ b_1,\ b_2,\ \cdots]$ are equal, then $a_n = b_n$ for all $n \ge 0$.
We restate the proof given in Burton pages 323-324.
*Proof*.
Let $x = [a_0;\ a_1,\ a_2,\ \cdots]$ then $C_0 < x < C_1$ which means that

$$a_0 < x < a_0 + \frac{1}{a_1}$$

We know that $a_1 \ge 1$ which gives the inequality

$$a_0 < x < a_0 + \frac{1}{a_1} < a_0 + 1$$

The greatest integer function $\lceil\ \rceil$ is

$$\lceil x \rceil = a_0$$

We are given that $[a_0;\ a_1,\ a_2,\ \cdots]$ and $[b_0;\ b_1,\ b_2,\ \cdots]$ are equal, so we have

$$a_0 + \frac{1}{[a_1;\ a_2,\ \cdots]} = x = b_0 + \frac{1}{[b_1;\ b_2,\ \cdots]} \qquad (*)$$

Therefore $a_0 = \lceil x \rceil = b_0$. By (*) we have

$$[a_1;\ a_2,\ \cdots] = [b_1;\ b_2,\ \cdots]$$

Repeating this argument gives

$$a_1 = b_1$$

Continuing this we have $a_2 = b_2$, $a_3 = b_3$, … By mathematical induction we have $a_n = b_n$ for all $n \ge 0$.

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