Complete Solutions to Supplementary Exercises on Integration

1. We are given $\int_0^{\frac{\pi}{2}} x\sin^2(x)\,\mathrm{d}x$. *How can we integrate this?*

   By applying integration by parts:

   $$\int (uv')\,\mathrm{d}x = uv - \int u'v\,\mathrm{d}x$$

   Let $u = x$ and $v' = \sin^2(x)$. Then

   $$u' = 1 \text{ and } v = \int \sin^2(x)\mathrm{d}x \underbrace{=}_{\substack{\text{By trig}\\\text{identity}}} \frac{1}{2}\int[1-\cos(2x)]\,\mathrm{d}x = \frac{1}{2}\left[x - \frac{\sin(2x)}{2}\right]$$

   Substituting this into the integration by parts formula gives

   $$\begin{aligned}\int_0^{\frac{\pi}{2}} x\sin^2(x)\,\mathrm{d}x &= [uv]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} u'v\,\mathrm{d}x \\ &= \left[x\frac{1}{2}\left(x - \frac{\sin(2x)}{2}\right)\right]_0^{\frac{\pi}{2}} - \frac{1}{2}\int_0^{\frac{\pi}{2}}\left[x - \frac{\sin(2x)}{2}\right]\mathrm{d}x \\ &= \frac{1}{2}\left\{\left[\frac{\pi}{2}\right]\left(\frac{\pi}{2} - \frac{\overbrace{\sin(\pi)}^{=0}}{2}\right) - \left[\frac{x^2}{2} + \frac{\cos(2x)}{4}\right]_0^{\frac{\pi}{2}}\right\} \quad \left[\text{Taking out } \frac{1}{2}\right] \\ &= \frac{1}{2}\left\{\left[\frac{\pi}{2}\right]\left(\frac{\pi}{2}\right) - \left[\frac{\pi^2}{8} + \frac{\overbrace{\cos(\pi)}^{=-1}}{4} - \left(0 + \frac{1}{4}\right)\right]\right\} \\ &= \frac{1}{2}\left\{\frac{\pi^2}{4} - \frac{\pi^2}{8} + \frac{1}{4} + \frac{1}{4}\right\} \\ &= \frac{1}{2}\left\{\frac{\pi^2}{8} + \frac{1}{2}\right\} = \frac{1}{4}\left(\frac{\pi^2}{4} + 1\right)\end{aligned}$$

2. We need to integrate $(\ln(x))^2$ between 1 and 2. We can write $(\ln(x))^2$ as

   $$(\ln(x))^2 = (\ln(x))^2 \times 1$$

   Again using integration by parts with $u = (\ln(x))^2$ and $v' = 1$:

   $$u' = 2\ln(x)\frac{1}{x} \text{ and } v = \int 1\mathrm{d}x = x$$

Putting this into the integration by parts formula gives

$$\begin{aligned}\int_1^2 \left(\ln(x)\right)^2 \,\mathrm{d}x &= \left[uv\right]_1^2 - \int_1^2 u'v \,\mathrm{d}x \\ &= \left[\left(\ln(x)\right)^2 x\right]_1^2 - \int_1^2 2\ln(x)\frac{1}{\cancel{x}}\cancel{x}\,\mathrm{d}x \\ &= \left[2\left(\ln(2)\right)^2 - \left(\ln(1)\right)^2\right] - 2\int_1^2 \ln(x)\,\mathrm{d}x \\ &= \left[2\left(\ln(2)\right)^2\right] - 2\int_1^2 \ln(x)\,\mathrm{d}x \qquad (\dagger)\end{aligned}$$

We need to find the last integral on the right hand side, $\int_1^2 \ln(x)\,\mathrm{d}x$:

Again applying integration by parts with

$$p = \ln(x) \text{ and } q' = 1$$

$$p' = \frac{1}{x} \text{ and } q = x$$

Therefore

$$\begin{aligned}\int_1^2 \ln(x)\,\mathrm{d}x &= \left[pq\right]_1^2 - \int_1^2 p'q\,\mathrm{d}x \\ &= \left[x\ln(x)\right]_1^2 - \int_1^2 \frac{1}{\cancel{x}}\cancel{x}\,\mathrm{d}x \\ &= \left[2\ln(2) - 0\right] - \left[x\right]_1^2 \qquad \left[\text{Because } \ln(1) = 0\right] \\ &= 2\ln(2) - 1\end{aligned}$$

Substituting this $\int_1^2 \ln(x)\,\mathrm{d}x = 2\ln(2) - 1$ into $(\dagger)$ gives

$$\begin{aligned}\int_1^2 \left(\ln(x)\right)^2 \mathrm{d}x &= \left[2\left(\ln(2)\right)^2\right] - 2\int_1^2 \ln(x)\,\mathrm{d}x \\ &= 2\left(\ln(2)\right)^2 - 2\left(2\ln(2) - 1\right) \\ &= 2\left[\left(\ln(2)\right)^2 - 2\ln(2) + 1\right] \qquad \left[\text{Taking out a factor of } 2\right] \\ &= 2\left[\left(\ln(2)\right) - 1\right]^2 \qquad \left[\text{By identity } a^2 - 2ab + b^2 = (a-b)^2\right]\end{aligned}$$

3. We are asked to find $\int_0^{\pi} x^3 \sin(x)\,dx$. We repeatedly use integration by parts:

$$\begin{aligned}\int_0^{\pi} \underbrace{x^3}_{=u} \underbrace{\sin(x)}_{=v'}\,dx &= \left[uv\right]_0^{\pi} - \int_0^{\pi} (u'v)\,dx \\ &= \left[-x^3\cos(x)\right]_0^{\pi} + 3\int_0^{\pi} \left(x^2\cos(x)\right)dx \\ &= \left[-\pi^3 \underbrace{\cos(\pi)}_{=-1} - 0\right] + 3\int_0^{\pi}\left(x^2\cos(x)\right)dx \\ &= \pi^3 + 3\int_0^{\pi}\left(x^2\cos(x)\right)dx \qquad (*)\end{aligned}$$

Applying integration by parts to find the last integral on the right hand side:

$$\begin{aligned}\int_0^{\pi}\left(x^2\cos(x)\right)dx &= \underbrace{\left[x^2\sin(x)\right]_0^{\pi}}_{=0} - \int_0^{\pi}\left(2x\sin(x)\right)dx \\ &= 0 - 2\int_0^{\pi} x\sin(x)\,dx \\ &= -2\left\{\left[-x\cos(x)\right]_0^{\pi} + \int_0^{\pi}\cos(x)\,dx\right\} \qquad \left[\text{Applying integration by parts again}\right] \\ &= -2\left\{\left[-\pi\underbrace{\cos(\pi)}_{=-1} - 0\right] + \underbrace{\left[\sin(x)\right]_0^{\pi}}_{=0}\right\} \\ &= -2\pi\end{aligned}$$

Putting this $\int_0^{\pi}\left(x^2\cos(x)\right)dx = -2\pi$ into (*) yields

$$\begin{aligned}\int_0^{\pi} x^3\sin(x)\,dx &= \pi^3 + 3\int_0^{\pi}\left(x^2\cos(x)\right)dx \\ &= \pi^3 + 3(-2\pi) = \pi^3 - 6\pi\end{aligned}$$

4. We are asked to compute $\int_0^{\frac{1}{\sqrt{2}}} \frac{x^3}{\left(1-x^2\right)^{\frac{3}{2}}}\,dx$. Using integration by parts formula with $u = x^2$ and $v' = \frac{x}{\left(1-x^2\right)^{3/2}}$. Then

$$u' = 2x \text{ and } v = \int \frac{x}{\left(1-x^2\right)^{3/2}}\,\mathrm{d}x$$

How do we find v?

By using integration by substitution with

$$p = 1 - x^2 \;\Rightarrow\; \frac{\mathrm{d}p}{\mathrm{d}x} = -2x \;\Rightarrow\; \mathrm{d}x = -\frac{\mathrm{d}p}{2x}$$

Therefore

$$v = \int \frac{x}{\left(1-x^2\right)^{3/2}}\,\mathrm{d}x$$

$$= -\int \frac{\cancel{x}}{p^{3/2}}\frac{\mathrm{d}p}{2\cancel{x}} = -\frac{1}{2}\int p^{-\frac{3}{2}}\mathrm{d}p = -\frac{1}{2}\frac{p^{-\frac{1}{2}}}{\left(-1/2\right)} = p^{-\frac{1}{2}} = \left(1-x^2\right)^{-\frac{1}{2}} = \frac{1}{\sqrt{1-x^2}}$$

Now applying the integration by parts formula gives

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{x^3}{\left(1-x^2\right)^{\frac{3}{2}}}\,\mathrm{d}x = \left[uv\right]_0^{\frac{1}{\sqrt{2}}} - \int_0^{\frac{1}{\sqrt{2}}} \left(u'v\right)\mathrm{d}x$$

$$= \left[x^2\frac{1}{\sqrt{1-x^2}}\right]_0^{\frac{1}{\sqrt{2}}} - \int_0^{\frac{1}{\sqrt{2}}}\left(2x\frac{1}{\sqrt{1-x^2}}\right)\mathrm{d}x$$

$$= \left[\frac{1}{2}\frac{1}{\sqrt{1-\frac{1}{2}}}\right] - 2\int_0^{\frac{1}{\sqrt{2}}}\left(x\frac{1}{\sqrt{1-x^2}}\right)\mathrm{d}x \qquad (\dagger)$$

The integral on the right hand side of (†) is calculated by substitution with $p = 1 - x^2$ then by using the above computation of v:

$$\int_0^{\frac{1}{\sqrt{2}}}\left(x\frac{1}{\sqrt{1-x^2}}\right)\mathrm{d}x = -\frac{1}{2}\int_1^{1/2} p^{-\frac{1}{2}}\,\mathrm{d}p = -\frac{\cancel{1}}{\cancel{2}}\left[\frac{p^{\frac{1}{2}}}{\cancel{1/2}}\right]_1^{\frac{1}{2}} = \frac{1}{\sqrt{2}} - 1$$

Putting this into (†) yields

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{x^3}{\left(1-x^2\right)^{\frac{3}{2}}}\,\mathrm{d}x = \left[\frac{1}{2}\frac{1}{\sqrt{1-\frac{1}{2}}}\right] + 2\left(\frac{1}{\sqrt{2}} - 1\right)$$

$$= \frac{\sqrt{2}}{2} + \frac{2}{\sqrt{2}} - 2 = \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} - 2 = \frac{3}{\sqrt{2}} - 2$$

5. *How do we calculate* $\int_0^3 \frac{x}{1+x^4}\,\mathrm{d}x$ *?*

Use integration by substitution with $u = x^2$; then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x \Rightarrow \mathrm{d}x = \frac{\mathrm{d}u}{2x}$$

Changing the limits of integration gives:

When $x = 0,\ u = 0$ and $x = 3,\ u = 3^2 = 9$. Using the limits we have

$$\int_0^3 \frac{x}{1+x^4}\,\mathrm{d}x = \int_0^9 \frac{\cancel{x}}{1+u^2}\frac{\mathrm{d}u}{2\cancel{x}} \underbrace{=}_{\text{by standard integral}} \frac{1}{2}\left[\tan^{-1}(u)\right]_0^9 = \frac{1}{2}\tan^{-1}(9)$$

6. We need to find $\int_1^e x^{-\frac{1}{2}}\ln(x)\,\mathrm{d}x$. Applying integration by parts with

$$u = \ln(x) \text{ and } v' = x^{-\frac{1}{2}}$$

Then $u' = \frac{1}{x}$ and $v = \int x^{-\frac{1}{2}}\mathrm{d}x = \frac{x^{\frac{1}{2}}}{1/2} = 2x^{\frac{1}{2}}$.

We have

$$\begin{aligned}\int_1^e x^{-\frac{1}{2}}\ln(x)\,\mathrm{d}x &= [uv]_1^e - \int_1^e (u'v)\,\mathrm{d}x \\ &= \left[2x^{\frac{1}{2}}\ln(x)\right]_1^e - 2\int_1^e \frac{1}{x}x^{\frac{1}{2}}\mathrm{d}x \\ &= \left[2e^{\frac{1}{2}}\underbrace{\ln(e)}_{=1} - 0\right] - 2\int_1^e x^{-\frac{1}{2}}\mathrm{d}x \\ &= 2e^{\frac{1}{2}} - 2\left[\frac{x^{\frac{1}{2}}}{1/2}\right]_1^e = 2e^{\frac{1}{2}} - 4\left[e^{\frac{1}{2}} - 1\right] = 4 - 2e^{\frac{1}{2}}\end{aligned}$$

7. *How do we evaluate* $\int_1^e \frac{1}{x\sqrt{1-[\ln(x)]^2}}\,\mathrm{d}x$ *?*

By substitution with $u = \ln(x)$. Differentiating this gives

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x} \Rightarrow \mathrm{d}x = x\,\mathrm{d}u$$

Our new limits are $\ln(1)=0$ and $\ln(e)=1$. We have

$$\int_1^e \frac{1}{x\sqrt{1-[\ln(x)]^2}}\,dx = \int_0^1 \frac{1}{\cancel{x}\sqrt{1-u^2}}\cancel{x}\,du$$

$$= \left[\sin^{-1}(u)\right]_0^1 = \sin^{-1}(1) = \frac{\pi}{2} \quad \left[\text{Because } \int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right)\right]$$

8. We need to find $\int_1^a \frac{e^{1/x}}{x^2}\,dx$. *How?*

By substitution with $u=\frac{1}{x}$. Differentiating this gives

$$\frac{du}{dx} = -x^{-2} = -\frac{1}{x^2} \quad \Rightarrow \quad dx = -x^2 du$$

Our limits are:

When $x=a$ then $u=\frac{1}{a}$ and when $x=1$ then $u=1$. We have

$$\int_1^a \frac{e^{1/x}}{x^2}\,dx = \int_1^{1/a} \frac{e^u}{\cancel{x^2}}\left(-\cancel{x^2}\right)du$$

$$= -\left[e^u\right]_1^{1/a} = -\left[e^{1/a}-e\right] = e - e^{\frac{1}{a}}$$

9. *How do we find* $\int_{-2}^2 \frac{1}{16-x^2}\,dx$ *?*

Use the standard integral given in the book:

$$(8.30) \qquad \int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right|$$

Applying this to the above gives

$$\int_{-2}^2 \frac{1}{16-x^2}\,dx = \int_{-2}^2 \frac{1}{4^2-x^2}\,dx$$

$$= \frac{1}{2\times 4}\left[\ln\left(\frac{4+x}{4-x}\right)\right]_{-2}^2$$

$$= \frac{1}{8}\left[\ln(3) - \ln\left(\frac{1}{3}\right)\right]$$

$$\underbrace{=}_{\substack{\text{Using} \\ \ln(a)-\ln(b)=\ln(a/b)}} \frac{1}{8}\ln\left(\frac{3}{1/3}\right) = \frac{1}{8}\ln\left(3^2\right) = \frac{2}{8}\ln(3) = \frac{1}{4}\ln(3)$$

10. *How do we calculate* $\int_1^3 \frac{1}{\sqrt{4x-x^2}}\,\mathrm{d}x$ *?*

By completing the square on the quadratic:

$$4x - x^2 = -x^2 + 4x$$
$$= -\left[x^2 - 4x\right] = -\left[(x-2)^2 - 2^2\right] = 2^2 - (x-2)^2$$

Substituting this $4x - x^2 = 2^2 - (x-2)^2$ into the above integrand gives

$$\int_1^3 \frac{1}{\sqrt{4x-x^2}}\,\mathrm{d}x = \int_1^3 \frac{1}{\sqrt{2^2-(x-2)^2}}\,\mathrm{d}x$$

But how do we integrate this $\int_1^3 \frac{1}{\sqrt{2^2-(x-2)^2}}\,\mathrm{d}x$ *?*

By applying the standard integral formula $\int \frac{1}{\sqrt{\alpha^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{\alpha}\right)$:

$$\int_1^3 \frac{1}{\sqrt{2^2-(x-2)^2}}\,\mathrm{d}x = \left[\sin^{-1}\left(\frac{x-2}{2}\right)\right]_1^3$$
$$= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3}$$

11. We need to find $\int_a^b \left\{(x-a)(b-x)\right\}^{-\frac{1}{2}}\mathrm{d}x$. This answer involves a lot of algebra, especially completing the square. We first examine $(x-a)(b-x)$:

$$(x-a)(b-x) = xb - x^2 - ab + ax = -x^2 + (a+b)x - ab$$

Completing the square on the last expression gives

$$
\begin{aligned}
-x^2+(a+b)x-ab &= -\left[x^2-(a+b)x+ab\right] \\
&= -\left[\left(x-\frac{a+b}{2}\right)^2+ab-\left(\frac{a+b}{2}\right)^2\right] \quad \text{[Completing the square]} \\
&= -\left[\left(x-\frac{a+b}{2}\right)^2+ab-\frac{a^2+2ab+b^2}{4}\right] \\
&= -\left[\left(x-\frac{a+b}{2}\right)^2+\frac{4ab-a^2-2ab-b^2}{4}\right] \\
&= -\left[\left(x-\frac{a+b}{2}\right)^2+\frac{-a^2+2ab-b^2}{4}\right] \\
&= -\left[\left(x-\frac{a+b}{2}\right)^2-\left(\frac{a^2-2ab+b^2}{4}\right)\right] \\
&= -\left[\left(x-\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2\right] \underbrace{=}_{\substack{\text{Taking the}\\ \text{minus sign in}}} \left(\frac{a-b}{2}\right)^2-\left(x-\frac{a+b}{2}\right)^2
\end{aligned}
$$

Now using the integral result $\int \frac{du}{\sqrt{\alpha^2-u^2}}=\sin^{-1}\left(\frac{u}{\alpha}\right)$ with

$$\alpha=\frac{a-b}{2} \quad \text{and} \quad u=x-\frac{a+b}{2}$$

We have

$$
\begin{aligned}
\int_a^b \left\{(x-a)(b-x)\right\}^{-\frac{1}{2}}\mathrm{d}x &= \int_a^b \frac{1}{\sqrt{(x-a)(b-x)}}\,\mathrm{d}x \\
&= \int_a^b \frac{1}{\sqrt{\left(\frac{a-b}{2}\right)^2-\left(x-\frac{a+b}{2}\right)^2}}\,\mathrm{d}x \quad \text{[From above]} \\
&= \left[\sin^{-1}\left(\frac{x-\frac{a+b}{2}}{\frac{a-b}{2}}\right)\right]_a^b \\
&= \left[\sin^{-1}\left(\frac{2x-(a+b)}{a-b}\right)\right]_a^b \quad \left[\begin{array}{l}\text{Multiplying numerator}\\ \text{and denominator by 2}\end{array}\right] \\
&= \sin^{-1}\left(\frac{b-a}{a-b}\right)-\sin^{-1}\left(\frac{a-b}{a-b}\right) \\
&= \sin^{-1}(-1)-\sin^{-1}(1)=\frac{3\pi}{2}-\frac{\pi}{2}=\pi
\end{aligned}
$$

12. We need to find $\int_{-\pi}^{\pi} \left|\cos(x)+\sin(x)\right| dx$. First converting the $\cos(x)+\sin(x)$ into amplitude phase form which is formula (4.75) in the book:

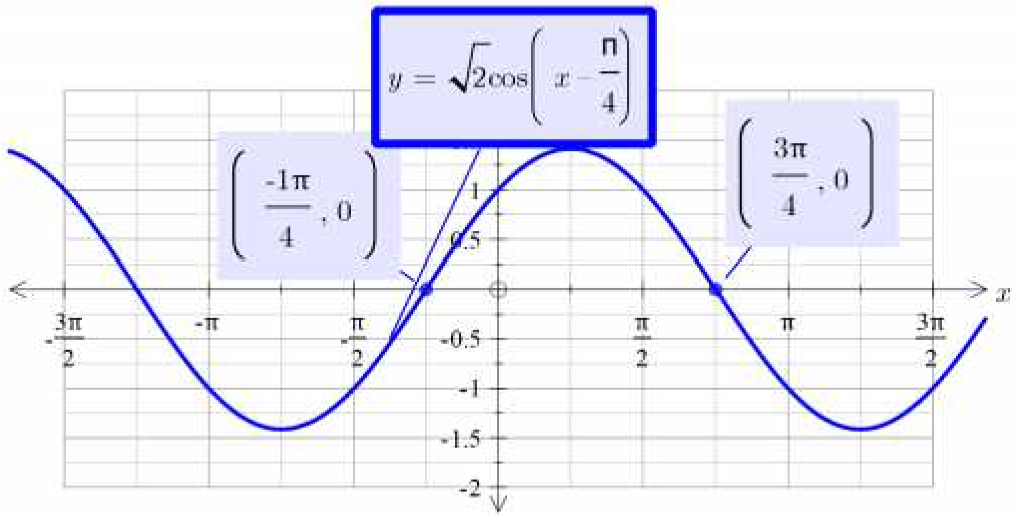
(4.75) $$a\cos(x)+b\sin(x)=r\cos(x-\beta)$$

where $r=\sqrt{a^2+b^2}$ and $\beta=\tan^{-1}\left(\frac{b}{a}\right)$

Applying this to $\cos(x)+\sin(x)$ gives

$$\cos(x)+\sin(x)=\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)$$

Since the integrand is the modulus of this so we need to be careful when this function $\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)$ is negative. Easier to see this in the sketch of the graph:



We split the integration between the positive and negative parts of the function because the modulus function definition is

$$|u|=\begin{cases} u & \text{if } u>0 \\ -u & \text{if } u<0 \end{cases}$$

Therefore

$$\int_{-\pi}^{\pi}\left|\cos(x)+\sin(x)\right|\,dx = \sqrt{2}\int_{-\pi}^{\pi}\left|\cos\left(x-\frac{\pi}{4}\right)\right|\,dx \qquad \left[\begin{array}{l}\text{Using the amplitude}\\ \text{phase form}\end{array}\right]$$

$$= \sqrt{2}\left[\int_{-\pi/4}^{3\pi/4}\cos\left(x-\frac{\pi}{4}\right)dx - \int_{3\pi/4}^{\pi}\cos\left(x-\frac{\pi}{4}\right)dx - \int_{-\pi}^{-\pi/4}\cos\left(x-\frac{\pi}{4}\right)dx\right]$$

$$= \sqrt{2}\left[\left[\sin\left(x-\frac{\pi}{4}\right)\right]_{-\pi/4}^{3\pi/4} - \left[\sin\left(x-\frac{\pi}{4}\right)\right]_{3\pi/4}^{\pi} - \left[\sin\left(x-\frac{\pi}{4}\right)\right]_{-\pi}^{-\pi/4}\right]$$

$$= \sqrt{2}\left[\left[\sin\left(\frac{\pi}{2}\right)-\sin\left(-\frac{\pi}{2}\right)\right]-\left[\sin\left(\frac{3\pi}{4}\right)-\sin\left(\frac{\pi}{2}\right)\right]-\left[\sin\left(-\frac{\pi}{2}\right)-\sin\left(-\frac{5\pi}{4}\right)\right]\right]$$

$$= \sqrt{2}\left[\left[1-(-1)\right]-\left[\frac{1}{\sqrt{2}}-1\right]-\left[-1-\frac{1}{\sqrt{2}}\right]\right] = 4\sqrt{2}$$

13. (a) We need to investigate the improper integral $\int_0^1 (1-x)\ln(x)\,dx$. First we find the indefinite integral by using integration by parts and then we place the integral limits.

Let $u=\ln(x)$ and $v'=1-x$. Then

$$u' = \frac{1}{x} \text{ and } v = \int (1-x)\,dx = x-\frac{x^2}{2}$$

Therefore we have

$$\int (1-x)\ln(x)\,dx = uv - \int u'v\,dx$$

$$= \left(x-\frac{x^2}{2}\right)\ln(x) - \int \frac{1}{x}\left(x-\frac{x^2}{2}\right)dx$$

$$= \left(x-\frac{x^2}{2}\right)\ln(x) - \int \left(1-\frac{x}{2}\right)dx$$

$$= \left(x-\frac{x^2}{2}\right)\ln(x) - \left[x-\frac{x^2}{4}\right]$$

Using the limits we have

$$\int_0^1 (1-x)\ln(x)\,dx = \lim_{c\to 0}\left[\left(x-\frac{x^2}{2}\right)\ln(x)-\left[x-\frac{x^2}{4}\right]\right]_c^1$$

$$= \left(1-\frac{1^2}{2}\right)\ln(1) - \left[1-\frac{1^2}{4}\right] - \lim_{c\to 0}\left[\left(c-\frac{c^2}{2}\right)\ln(c)-\left[c-\frac{c^2}{4}\right]\right]$$

$$= 0-\frac{3}{4}-\lim_{c\to 0}\left(c-\frac{c^2}{2}\right)\ln(c)+\underbrace{\lim_{c\to 0}\left(c-\frac{c^2}{4}\right)}_{=0}$$

$$= -\frac{3}{4}-\lim_{c\to 0}\left(c-\frac{c^2}{2}\right)\ln(c) \qquad (\dagger)$$

We need to find the limit $\lim_{c\to 0}\left(c-\frac{c^2}{2}\right)\ln\left(c\right)$ in (†). *How do we evaluate this limit?*

We know the series expansion of $\ln\left(1+x\right)$ which is given in chapter 7 and is

(7.21) $\quad \ln\left(1+x\right)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$ provided $-1<x\le 1$

If we substitute $y=x-1$ into this formula (7.21) then we get the Taylor series around 1:

$$\ln\left(y\right)=\left(y-1\right)-\frac{\left(y-1\right)^2}{2}+\frac{\left(y-1\right)^3}{3}-\frac{\left(y-1\right)^4}{4}+\cdots \text{ provided } \quad 0<y\le 2$$

Using this and evaluating the above limit $\lim_{c\to 0}\left(c-\frac{c^2}{2}\right)\ln\left(c\right)$:

$$\begin{aligned}\lim_{c\to 0}\left(c-\frac{c^2}{2}\right)\ln\left(c\right)&=\lim_{c\to 0}\left[c\left(1-\frac{c}{2}\right)\ln\left(c\right)\right]\\&=\lim_{c\to 0}\left[c\left(1-\frac{c}{2}\right)\left(\left(c-1\right)-\frac{\left(c-1\right)^2}{2}+\frac{\left(c-1\right)^3}{3}-\frac{\left(c-1\right)^4}{4}+\cdots\right)\right]=0\end{aligned}$$

Putting this into (†) gives

$$\int_0^1\left(1-x\right)\ln\left(x\right)\,\mathrm{d}x=-\frac{3}{4}-\lim_{c\to 0}\left(c-\frac{c^2}{2}\right)\ln\left(c\right)=-\frac{3}{4}-0=-\frac{3}{4}$$

(b) We first find the integral and then substitute the limits:

$$\int x^3e^{-x^2}\,\mathrm{d}x=\int x^2\cdot xe^{-x^2}\,\mathrm{d}x$$

We use integration by parts to evaluate this integral with

$$u=x^2 \text{ and } v'=xe^{-x^2}$$

Then $u'=2x$ and

$$v=\int xe^{-x^2}\,\mathrm{d}x \underbrace{=}_{\substack{\text{Letting } p=x^2\\ \text{then } \frac{\mathrm{d}p}{\mathrm{d}x}=2x}} \int \cancel{x}e^{-p}\frac{\mathrm{d}p}{2\cancel{x}}=-\frac{1}{2}e^{-p}=-\frac{1}{2}e^{-x^2}$$

Putting this into the integration by parts formula gives

$$\begin{aligned}\int x^3 e^{-x^2}\,\mathrm{d}x &= uv - \int u'v\,\mathrm{d}x\\ &= x^2\left(-\frac{1}{2}e^{-x^2}\right) + \frac{1}{2}\int 2xe^{-x^2}\,\mathrm{d}x\\ &= -\frac{1}{2}x^2e^{-x^2} + \int xe^{-x^2}\,\mathrm{d}x\\ &= -\frac{1}{2}x^2e^{-x^2} - \frac{1}{2}e^{-x^2} \qquad \left[\text{From above } \int xe^{-x^2}\,\mathrm{d}x = -\frac{1}{2}e^{-x^2}\right]\\ &= -\frac{1}{2}e^{-x^2}\left(x^2+1\right)\end{aligned}$$

Sticking in the limits gives

$$\begin{aligned}\int_0^{+\infty} x^3e^{-x^2}\,\mathrm{d}x &= \lim_{M\to\infty}\left[-\frac{1}{2}e^{-x^2}\left(x^2+1\right)\right]_0^M\\ &= \lim_{M\to\infty}\left[-\frac{1}{2}e^{-M^2}\left(M^2+1\right)\right] - \left[-\frac{1}{2}e^0\left(0+1\right)\right]\\ &= \lim_{M\to\infty}\left[-\frac{1}{2}e^{-M^2}\left(M^2+1\right)\right] + \frac{1}{2} \qquad (*)\end{aligned}$$

Using the power series expansion of e^x which is formula (7.15)

(7.15) $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

For e^{x^2} we have

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots \qquad (**)$$

Using this in the evaluation of the limit in (*) yields

$$\begin{aligned}\lim_{M\to\infty}\left[-\frac{1}{2}e^{-M^2}\left(M^2+1\right)\right] &= \lim_{M\to\infty}\left[-\frac{1}{2}\frac{\left(M^2+1\right)}{e^{M^2}}\right]\\ &= \lim_{M\to\infty}\left(-\frac{1}{2}\frac{M^2+1}{1+M^2+\frac{M^4}{2!}+\frac{M^6}{3!}+\cdots}\right) \qquad \left[\text{By } (**)\right]\\ &= \lim_{M\to\infty}\left(-\frac{1}{2}\frac{1+1/M^2}{\frac{1}{M^2}+1+\frac{M^2}{2!}+\frac{M^3}{3!}+\cdots}\right) \qquad \left[\begin{array}{l}\text{Divide numerator and}\\ \text{denominator by } M^2\end{array}\right]\\ &= -\frac{1}{2}\left[\frac{1+0}{0+1+\lim_{M\to\infty}\left(\frac{M^2}{2!}+\frac{M^3}{3!}+\cdots\right)}\right] = -\frac{1}{2}\left(0\right) = 0\end{aligned}$$

Putting this into (*) gives

$$\int_0^{+\infty} x^3 e^{-x^2} \mathrm{d}x = \lim_{M\to\infty}\left[-\frac{1}{2}e^{-M^2}\left(M^2+1\right)\right]+\frac{1}{2} = 0+\frac{1}{2} = \frac{1}{2}$$

(c) We need to examine the improper integral $\int_0^{+\infty} \frac{\tan^{-1}(x)}{x^2}\mathrm{d}x$.

Again we first evaluate the integral without the limits (indefinite integral). We have

$$\int \frac{\tan^{-1}(x)}{x^2}\,\mathrm{d}x = \int x^{-2}\tan^{-1}(x)\,\mathrm{d}x$$

We use integration by parts with $u = \tan^{-1}(x)$ and $v' = x^{-2}$. Differentiating one and integrating the other gives

$$u' = \frac{1}{1+x^2} \quad \text{and} \quad v = \int x^{-2}\mathrm{d}x = -x^{-1} = -\frac{1}{x}$$

Putting these into the formula gives

$$\begin{aligned}\int \frac{\tan^{-1}(x)}{x^2}\,\mathrm{d}x &= uv - \int u'v\,\mathrm{d}x \\ &= -\frac{1}{x}\tan^{-1}(x) + \int \frac{1}{1+x^2}\frac{1}{x}\,\mathrm{d}x \qquad (\dagger)\end{aligned}$$

We need to find the integral on the right hand side of (†) by using partial fractions:

$$\frac{1}{1+x^2}\frac{1}{x} = \frac{1}{x\left(1+x^2\right)} = \frac{A}{x}+\frac{Bx+C}{1+x^2}$$

From this we have

$$1 = A\left(1+x^2\right)+\left(Bx+C\right)x \qquad (*)$$

Substituting $x=0$ into (*) yields

$$1 = A$$

Equating coefficients of x^2 in (*):

$$0 = A+B = 1+B \Rightarrow B=-1$$

Equating coefficients of x in (*):

$$0 = 0+C \Rightarrow C=0$$

Substituting these values of A, B and C into the above gives:

$$\frac{1}{1+x^2}\frac{1}{x} = \frac{1}{x}-\frac{x}{1+x^2}$$

Now the integral on the right of (†) is easy to evaluate

$$\int \frac{1}{1+x^2}\frac{1}{x}\,\mathrm{d}x = \int \frac{1}{x}\mathrm{d}x - \int \frac{x}{1+x^2}\,\mathrm{d}x = \ln\left(x\right) - \frac{1}{2}\ln\left(1+x^2\right) = \ln\left(\frac{x}{\left(1+x^2\right)^{\frac{1}{2}}}\right)$$

Putting this into (†) gives

$$\int \frac{\tan^{-1}\left(x\right)}{x^2}\,\mathrm{d}x = -\frac{1}{x}\tan^{-1}\left(x\right) + \ln\left(\frac{x}{\sqrt{1+x^2}}\right)$$

Splitting the given integral between 0 to 1 and 1 to $+\infty$ we have:

$$\int_0^{+\infty} \frac{\tan^{-1}\left(x\right)}{x^2}\mathrm{d}x = \int_0^1 \frac{\tan^{-1}\left(x\right)}{x^2}\mathrm{d}x + \int_1^{+\infty} \frac{\tan^{-1}\left(x\right)}{x^2}\mathrm{d}x$$
$$= \lim_{c\to 0}\int_0^1 \frac{\tan^{-1}\left(x\right)}{x^2}\mathrm{d}x + \lim_{M\to+\infty}\int_1^M \frac{\tan^{-1}\left(x\right)}{x^2}\mathrm{d}x$$

Working through each of these integrals separately:

$$\lim_{c\to 0}\int_c^1 \frac{\tan^{-1}\left(x\right)}{x^2}\mathrm{d}x = \lim_{c\to 0}\left[-\frac{1}{x}\tan^{-1}\left(x\right) + \ln\left(\frac{x}{\sqrt{1+x^2}}\right)\right]_c^1 \qquad \left[\text{From above}\right]$$
$$= \left[-\frac{1}{1}\tan^{-1}\left(1\right) + \ln\left(\frac{1}{\sqrt{1+1^2}}\right)\right] - \lim_{c\to 0}\left[-\frac{1}{c}\tan^{-1}\left(c\right) + \ln\left(\frac{c}{\sqrt{1+c^2}}\right)\right]$$
$$= -\frac{\pi}{4} + \ln\left(\frac{1}{\sqrt{2}}\right) - \lim_{c\to 0}\left[-\frac{1}{c}\tan^{-1}\left(c\right) + \ln\left(\frac{c}{\sqrt{1+c^2}}\right)\right]$$

Now

$$\lim_{c\to 0}\left[\ln\left(\frac{c}{\sqrt{1+c^2}}\right)\right] = \lim_{c\to 0}\left[\ln\left(c\right) - \frac{1}{2}\ln\left(1+c^2\right)\right] = -\infty$$

Since one of the integral diverges so the given integral $\int_0^{+\infty} \frac{\tan^{-1}\left(x\right)}{x^2}\mathrm{d}x$ diverges.

14. The shaded area is given by the integral

$$\int_{\frac{\pi}{2}}^{\pi} \frac{1}{\sqrt{1-\cos\left(x\right)}}\,\mathrm{d}x$$

We can rewrite the term under the square root sign as

$$1-\cos\left(x\right) = \cos\left(0\right) - \cos\left(x\right)$$

Applying the trigonometric identity:

(4.63) $$\cos(A)-\cos(B)=2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$$

On $\cos(0)-\cos(x)$ gives

$$\cos(0)-\cos(x)=2\sin\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)=2\sin^2\left(\frac{x}{2}\right)$$

Taking the square root of this yields

$$\sqrt{1-\cos(x)}=\sqrt{2\sin^2\left(\frac{x}{2}\right)}=\sqrt{2}\sin\left(\frac{x}{2}\right)$$

Substituting this $\sqrt{1-\cos(x)}=\sqrt{2}\sin\left(\frac{x}{2}\right)$ into the given integral:

$$\int_{\frac{\pi}{2}}^{\pi}\frac{1}{\sqrt{1-\cos(x)}}\,\mathrm{d}x=\frac{1}{\sqrt{2}}\int_{\frac{\pi}{2}}^{\pi}\frac{1}{\sin(x/2)}\,\mathrm{d}x \qquad (\dagger)$$

Using the trigonometric identity

(4.72) $$\sin(x)=\frac{2t}{1+t^2}\quad\text{where}\quad t=\tan\left(\frac{x}{2}\right)$$

We have

$$\sin\left(\frac{x}{2}\right)=\frac{2t}{1+t^2}\quad\text{where}\quad t=\tan\left(\frac{x}{4}\right)$$

Differentiating $t=\tan\left(\frac{x}{4}\right)$ gives

$$\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{1}{4}\sec^2\left(\frac{x}{4}\right)=\frac{1}{4}\left[1+\tan^2\left(\frac{x}{4}\right)\right]=\frac{1}{4}\left[1+t^2\right]$$

We have $\mathrm{d}x=\frac{4\mathrm{d}t}{1+t^2}$. Changing the limits of integration:

When $x=\frac{\pi}{2}$ then $\tan\left(\frac{\pi/2}{4}\right)=\tan\left(\frac{\pi}{8}\right)$ and at $x=\pi$ then $\tan\left(\frac{\pi}{4}\right)=1$.

Using this substitution in (†) gives

$$\frac{1}{\sqrt{2}}\int_{\frac{\pi}{2}}^{\pi}\frac{1}{\sin(x/2)}\,\mathrm{d}x=\frac{1}{\sqrt{2}}\int_{\tan\left(\frac{\pi}{8}\right)}^{1}\frac{\cancel{1+t^2}}{2t}\,\frac{4\mathrm{d}t}{\cancel{1+t^2}}$$

$$=\frac{2}{\sqrt{2}}\int_{\tan\left(\frac{\pi}{8}\right)}^{1}\frac{\mathrm{d}t}{t}$$

$$=\sqrt{2}\left[\ln(t)\right]_{\tan\left(\frac{\pi}{8}\right)}^{1}=\sqrt{2}\left[\ln(1)-\ln\left(\tan\left(\frac{\pi}{8}\right)\right)\right]=-\sqrt{2}\left[\ln\left(\tan\left(\frac{\pi}{8}\right)\right)\right]$$

15. (a) Let $u=1+x^2$ then $\frac{\mathrm{d}u}{\mathrm{d}x}=2x \Rightarrow \mathrm{d}x=\frac{\mathrm{d}u}{2x}$ and the given integral without the limits is

$$\int\frac{x^3}{\left(x^2+1\right)^3}\,\mathrm{d}x=\int\frac{x}{\left(x^2+1\right)^3}x^2\mathrm{d}x$$

$$=\int\frac{\cancel{x}}{u^3}x^2\,\frac{\mathrm{d}u}{2\cancel{x}}$$

$$=\frac{1}{2}\int\frac{x^2}{u^3}\,\mathrm{d}u$$

$$=\frac{1}{2}\int\frac{u-1}{u^3}\,\mathrm{d}u \qquad \left[\text{Because } u=1+x^2 \Rightarrow x^2=u-1\right]$$

$$=\frac{1}{2}\int\left(u^{-2}-u^{-3}\right)\mathrm{d}u$$

$$=\frac{1}{2}\left[-u^{-1}-\frac{u^{-2}}{-2}\right]=-\frac{1}{2}\left[\frac{1}{u}-\frac{1}{2u^2}\right]=-\frac{1}{2}\left[\frac{2u}{2u^2}-\frac{1}{2u^2}\right]=-\frac{1}{4}\left[\frac{2u-1}{u^2}\right]$$

Substituting back the given substitution $u=1+x^2$ into the above gives

$$\int\frac{x^3}{\left(x^2+1\right)^3}\,\mathrm{d}x=-\frac{1}{4}\left[\frac{2u-1}{u^2}\right]=-\frac{1}{4}\left[\frac{2\left(x^2+1\right)-1}{\left(x^2+1\right)^2}\right]=-\frac{1}{4}\left[\frac{2x^2+1}{\left(x^2+1\right)^2}\right] \qquad (**)$$

Sticking in the limits yields

$$\int_0^a\frac{x^3}{\left(x^2+1\right)^3}\,\mathrm{d}x=-\frac{1}{4}\left[\frac{2x^2+1}{\left(x^2+1\right)^2}\right]_0^a$$

$$=-\frac{1}{4}\left[\frac{2a^2+1}{\left(a^2+1\right)^2}-1\right]$$

$$=-\frac{1}{4}\left[\frac{2a^2+1-\left(a^2+1\right)^2}{\left(a^2+1\right)^2}\right]$$

$$=-\frac{1}{4}\left[\frac{2a^2+1-a^4-2a^2-1}{\left(a^2+1\right)^2}\right]=-\frac{1}{4}\left[\frac{-a^4}{\left(a^2+1\right)^2}\right]=\frac{1}{4}\left[\frac{a^4}{\left(a^2+1\right)^2}\right]=\frac{a^4}{\left[2\left(a^2+1\right)\right]^2}$$

(b) This time we use the substitution $u = \tan^{-1}(x)$. Differentiating this gives

$$\frac{du}{dx} = \frac{1}{1+x^2} \quad \Rightarrow \quad dx = \left(1+x^2\right)du$$

Also from the given substitution $u = \tan^{-1}(x)$ we have

$$\tan(u) = x$$

The integral without the limits is equal to

$$\int \frac{x^3}{\left(x^2+1\right)^3}\,dx = \int \frac{\tan^3(u)}{\cancel{\left(x^2+1\right)}\left(\tan^2(u)+1\right)^2}\cancel{\left(1+x^2\right)}du = \int \frac{\tan(u)}{\left(\tan^2(u)+1\right)^2}\tan^2(u)\,du$$

To find the integral on the right hand side of the above expression we use integration by substitution:

Let $p = 1+\tan^2(u)$ then

$$\frac{dp}{du} = 2\tan(u)\sec^2(u) \underbrace{=}_{\text{Because } \sec^2(u)=1+\tan^2(u)} 2\tan(u)\underbrace{\left[1+\tan^2(u)\right]}_{=p} = 2\tan(u)p$$

We have $du = \dfrac{dp}{2\tan(u)p}$. Substituting these into the above evaluation

$\displaystyle\int \frac{\tan(u)}{\left(\tan^2(u)+1\right)^2}\tan^2(u)\,du$ yields

$$\int \frac{\tan(u)}{\left(\tan^2(u)+1\right)^2}\tan^2(u)\,du = \int \frac{\cancel{\tan(u)}}{p^2}(p-1)\frac{dp}{2\cancel{\tan(u)}p} \quad \begin{bmatrix}\text{Because} \\ \tan^2(u) = p-1\end{bmatrix}$$

$$= \frac{1}{2}\int \frac{p-1}{p^3}\,dp$$

$$= \frac{1}{2}\int \left(p^{-2}-p^{-3}\right)dp$$

$$= \frac{1}{2}\left[-p^{-1} - \frac{p^{-2}}{-2}\right]$$

$$= -\frac{1}{2}\left[\frac{1}{p} - \frac{1}{2p^2}\right] = -\frac{1}{2}\left[\frac{2p-1}{2p^2}\right] = -\frac{1}{4}\left[\frac{2p-1}{p^2}\right] \qquad (*)$$

Note that $p = 1+\tan^2(u)$ and $u = \tan^{-1}(x)$ therefore

$$p = 1 + \left[\tan\left(\tan^{-1}(x)\right)\right]^2 = 1+x^2$$

Putting this $p = 1+\tan^2(u) = 1+x^2$ into the above $(*)$ we have

$$\int \frac{x^3}{\left(x^2+1\right)^3}\,dx = \int \frac{\tan\left(u\right)}{\left(\tan^2\left(u\right)+1\right)^2}\tan^2\left(u\right)du$$
$$= -\frac{1}{4}\left[\frac{2\left(x^2+1\right)-1}{\left(x^2+1\right)^2}\right] = -\frac{1}{4}\left[\frac{2x^2+1}{\left(x^2+1\right)^2}\right]$$

This expression is identical to the one in (**) in part (a). Hence from part (a) we have

$$\int_0^a \frac{x^3}{\left(x^2+1\right)^3}\,dx = \frac{a^4}{\left[2\left(a^2+1\right)\right]^2}$$

16. We are given $I_n = \int_0^{\frac{\pi}{4}} \tan^n\left(x\right)dx$ and we need to show:

$$I_n + I_{n-2} = \frac{1}{n-1}$$

Working with I_n we have

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n\left(x\right)dx = \int_0^{\frac{\pi}{4}} \tan^{n-2}\left(x\right)\tan^2\left(x\right)dx$$
$$= \int_0^{\frac{\pi}{4}} \tan^{n-2}\left(x\right)\left[\sec^2\left(x\right)-1\right]dx \quad \left[\text{Because } \sec^2\left(x\right)-1=\tan^2\left(x\right)\right]$$
$$= \int_0^{\frac{\pi}{4}} \tan^{n-2}\left(x\right)\sec^2\left(x\right)dx - \underbrace{\int_0^{\frac{\pi}{4}} \tan^{n-2}\left(x\right)dx}_{=I_{n-2}} = \int_0^{\frac{\pi}{4}} \tan^{n-2}\left(x\right)\sec^2\left(x\right)dx - I_{n-2}$$

Adding I_{n-2} to both sides gives

$$I_n + I_{n-2} = \int_0^{\frac{\pi}{4}} \tan^{n-2}\left(x\right)\sec^2\left(x\right)dx \qquad (*)$$

Let $u=\tan\left(x\right)$ then differentiating this gives

$$\frac{du}{dx} = \sec^2\left(x\right) \Rightarrow dx = \frac{du}{\sec^2\left(x\right)}$$

The new limits are $u=\tan\left(\frac{\pi}{4}\right)=1$ and $u=\tan\left(0\right)=0$. Therefore the integral on the right hand side of (*) is given by

$$I_n + I_{n-2} = \int_0^{\frac{\pi}{4}} \tan^{n-2}(x)\sec^2(x)\,\mathrm{d}x$$
$$= \int_0^1 u^{n-2} \cancel{\sec^2(x)} \frac{\mathrm{d}u}{\cancel{\sec^2(x)}} = \int_0^1 u^{n-2}\mathrm{d}u = \left[\frac{u^{n-1}}{n-1}\right]_0^1 = \frac{1^{n-1}}{n-1} - 0 = \frac{1}{n-1}$$

Hence we have our required result; $I_n + I_{n-2} = \dfrac{1}{n-1}$.

17. We are asked to show

$$\int_0^{+\infty} x^n e^{-x^2}\mathrm{d}x = \frac{1}{2}(n-1)\int_0^{+\infty} x^{n-2}e^{-x^2}\mathrm{d}x$$

How do we prove this result?

By rewriting the integrand and then applying integration by parts:

$$\int_0^{+\infty} x^n e^{-x^2}\mathrm{d}x = \lim_{M\to+\infty}\int_0^M x^n e^{-x^2}\mathrm{d}x = \lim_{M\to+\infty}\int_0^M x^{n-1}\left(xe^{-x^2}\right)\mathrm{d}x$$

Let $u = x^{n-1}$ and $v' = xe^{-x^2}$. Differentiating u and integrating v' we have

$$u' = (n-1)x^{n-2} \text{ and } v = \int xe^{-x^2}\mathrm{d}x$$

In order to find v we use substitution with $p = x^2$ therefore

$$\frac{\mathrm{d}p}{\mathrm{d}x} = 2x \quad \Rightarrow \quad \mathrm{d}x = \frac{\mathrm{d}p}{2x}$$

We have

$$v = \int xe^{-x^2}\mathrm{d}x = \int \cancel{x}e^{-p}\frac{\mathrm{d}p}{2\cancel{x}} = \frac{1}{2}\int e^{-p}\mathrm{d}p = -\frac{1}{2}e^{-p} = -\frac{1}{2}e^{-x^2}$$

Substituting these evaluations into the integration by parts formula but without the limits gives

$$\int x^{n-1}\left(xe^{-x^2}\right)\mathrm{d}x = uv - \int u'v\,\mathrm{d}x$$
$$= -\frac{1}{2}x^{n-1}e^{-x^2} + \frac{1}{2}\int (n-1)x^{n-2}e^{-x^2}\mathrm{d}x$$
$$= -\frac{1}{2}x^{n-1}e^{-x^2} + \frac{(n-1)}{2}\int x^{n-2}e^{-x^2}\mathrm{d}x$$

Sticking in the limits we have

$$\lim_{M\to+\infty}\int_0^M x^{n-1}\left(xe^{-x^2}\right)\mathrm{d}x = \lim_{M\to+\infty}\left[-\frac{1}{2}x^{n-1}e^{-x^2}\right]_0^M + \frac{(n-1)}{2}\lim_{M\to+\infty}\int_0^M x^{n-2}e^{-x^2}\mathrm{d}x$$

$$= -\frac{1}{2}\left[\underbrace{\lim_{M\to+\infty}\left(M^{n-1}e^{-M^2}\right)}_{=0\text{ by Taylor series expansion}} - 0\right] + \frac{(n-1)}{2}\int_0^{+\infty} x^{n-2}e^{-x^2}\mathrm{d}x$$

$$= \frac{(n-1)}{2}\int_0^{+\infty} x^{n-2}e^{-x^2}\mathrm{d}x$$

Hence we have our required result, $\int_0^{+\infty} x^n e^{-x^2}\mathrm{d}x = \frac{1}{2}(n-1)\int_0^{+\infty} x^{n-2}e^{-x^2}\mathrm{d}x$.

We are asked to evaluate $\int_0^{+\infty} x^5 e^{-x^2}\mathrm{d}x$. Putting $n=5$ into the given result:

$$\int_0^{+\infty} x^5 e^{-x^2}\mathrm{d}x = \frac{1}{2}(5-1)\int_0^{+\infty} x^{5-2}e^{-x^2}\mathrm{d}x = 2\int_0^{+\infty} x^3 e^{-x^2}\mathrm{d}x \qquad (*)$$

We work out the last integral on the right hand side of (*) by applying the given formula again but with $n=3$:

$$\int_0^{+\infty} x^3 e^{-x^2}\mathrm{d}x = \frac{1}{2}(3-1)\int_0^{+\infty} x^{3-2}e^{-x^2}\mathrm{d}x = \int_0^{+\infty} xe^{-x^2}\mathrm{d}x$$

In the above we have already found the indefinite integral of the last integral. We had $\int xe^{-x^2}\mathrm{d}x = -\frac{1}{2}e^{-x^2}$. Substituting the limits we have

$$\int_0^{+\infty} x^3 e^{-x^2}\mathrm{d}x = \int_0^{+\infty} xe^{-x^2}\mathrm{d}x$$

$$= \lim_{M\to+\infty}\left[-\frac{1}{2}e^{-x^2}\right]_0^M = -\frac{1}{2}\left[\lim_{M\to+\infty}\left(e^{-M^2}\right) - e^0\right] = -\frac{1}{2}[0-1] = \frac{1}{2}$$

Substituting this $\int_0^{+\infty} x^3 e^{-x^2}\mathrm{d}x = \frac{1}{2}$ into (*) yields

$$\int_0^{+\infty} x^5 e^{-x^2}\mathrm{d}x = 2\int_0^{+\infty} x^3 e^{-x^2}\mathrm{d}x = 2\left(\frac{1}{2}\right) = 1$$

18. We need to find $I_n - I_{n-2}$ given $I_n = \int_0^{\frac{\pi}{2}} \frac{\cos(nx)-1}{\sin(x)}\mathrm{d}x$. Writing out $I_n - I_{n-2}$:

$$\begin{aligned}I_n - I_{n-2} &= \int_0^{\frac{\pi}{2}} \frac{\cos(nx)-1}{\sin(x)}\,\mathrm{d}x - \int_0^{\frac{\pi}{2}} \frac{\cos((n-2)x)-1}{\sin(x)}\,\mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \frac{\cos(nx)-1-\cos((n-2)x)+1}{\sin(x)}\,\mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \frac{\cos(nx)-\cos((n-2)x)}{\sin(x)}\,\mathrm{d}x\end{aligned}$$

Applying the trigonometric formula of chapter 4:

(4.63) $$\cos(A)-\cos(B) = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$$

To the numerator part of integrand we have

$$\begin{aligned}\cos(nx)-\cos((n-2)x) &= 2\sin\left(\frac{nx+nx-2x}{2}\right)\sin\left(\frac{nx-2x-nx}{2}\right) \\ &= 2\sin\left(\frac{\cancel{2}nx-\cancel{2}x}{\cancel{2}}\right)\sin(-x) \\ &= -2\sin[(n-1)x]\sin(x) \qquad [\text{Because } \sin(-x)=-\sin(x)]\end{aligned}$$

Substituting this into the numerator of the above integrand gives

$$\begin{aligned}I_n - I_{n-2} &= \int_0^{\frac{\pi}{2}} \frac{\cos(nx)-\cos((n-2)x)}{\sin(x)}\,\mathrm{d}x \\ &= -\int_0^{\frac{\pi}{2}} \frac{2\sin[(n-1)x]\,\cancel{\sin(x)}}{\cancel{\sin(x)}}\,\mathrm{d}x \\ &= -2\int_0^{\frac{\pi}{2}} \sin[(n-1)x]\,\mathrm{d}x \\ &= -2\left[-\frac{\cos[(n-1)x]}{n-1}\right]_0^{\frac{\pi}{2}} = \frac{2}{n-1}\left[\cos\left[(n-1)\frac{\pi}{2}\right]-1\right]\end{aligned}$$

We have the general formula $I_n - I_{n-2} = \frac{2}{n-1}\left[\cos\left[(n-1)\frac{\pi}{2}\right]-1\right]$.

Substituting $n=3$ into this in order to evaluate I_3 gives

$$I_3 - I_1 = \frac{2}{3-1}\left[\cos\left[(3-1)\frac{\pi}{2}\right]-1\right] = \left[\underbrace{\cos(\pi)}_{=-1}-1\right] = -2$$

Therefore by adding I_1 to both sides of the above we have

$$I_3 = -2 + I_1 \qquad (*)$$

To find I_3 we need to evaluate I_1. *What is* I_1 *equal to?*

Substituting $n = 1$ into the given formula; $I_n = \int_0^{\frac{\pi}{2}} \frac{\cos(nx) - 1}{\sin(x)}\,dx$:

$$I_1 = \int_0^{\frac{\pi}{2}} \frac{\cos(x) - 1}{\sin(x)}\,dx \qquad (\dagger)$$

Let $u = \cos(x) - 1$ then

$$\frac{du}{dx} = -\sin(x) \Rightarrow dx = -\frac{du}{\sin(x)}$$

The new limits are

$$u = \cos\left(\frac{\pi}{2}\right) - 1 = 0 - 1 = -1 \text{ and } u = \cos(0) - 1 = 1 - 1 = 0$$

We have

$$\begin{aligned} I_1 &= \int_0^{\frac{\pi}{2}} \frac{\cos(x) - 1}{\sin(x)}\,dx \\ &= -\int_0^{-1} \frac{u}{\sin(x)} \frac{du}{\sin(x)} = \int_{-1}^{0} \frac{u\,du}{\sin^2(x)} \qquad (\dagger\dagger) \end{aligned}$$

From the fundamental trigonometric identity

$$\sin^2(\theta) + \cos^2(\theta) = 1 \text{ we have } \sin^2(\theta) = 1 - \cos^2(\theta)$$

From above we have $u = \cos(x) - 1$ so $u + 1 = \cos(x)$ and

$$\sin^2(x) = 1 - \cos^2(x) = 1 - (u+1)^2 = 1 - u^2 - 2u - 1 = -u(u+2)$$

Substituting this into the above I_1 integral in $(\dagger\dagger)$ gives

$$\begin{aligned} I_1 = \int_{-1}^{0} \frac{u\,du}{\sin^2(x)} &= \int_{-1}^{0} \frac{\not{u}\,du}{-\not{u}(u+2)} \\ &= -\left[\ln(u+2)\right]_{-1}^{0} = -\left[\ln(2) - \ln(1)\right] = -\ln(2) \end{aligned}$$

We have $I_1 = \int_0^{\frac{\pi}{2}} \frac{\cos(x) - 1}{\sin(x)}\,dx = -\ln(2)$. Substituting this into (*) yields

$$I_3 = -2 + I_1 = -2 - \ln(2)$$

For I_4 we substitute $n = 4$ into the above derived formula,

$$I_n - I_{n-2} = \frac{2}{n-1}\left[\cos\left[\left(n-1\right)\frac{\pi}{2}\right] - 1\right]$$

To give

$$I_4 - I_2 = \frac{2}{4-1}\left[\cos\left[\left(4-1\right)\frac{\pi}{2}\right] - 1\right] = \frac{2}{3}\left[\cos\left(\frac{3\pi}{2}\right) - 1\right] = \frac{2}{3}\left[\underbrace{\cos\left(\frac{3\pi}{2}\right)}_{=0} - 1\right] = -\frac{2}{3}$$

Adding I_2 to both sides yields

$$I_4 = I_2 - \frac{2}{3} \qquad (**)$$

This time we need to evaluate I_2. *What is* I_2 *equal to?*

Substituting $n = 2$ into the given formula $I_n = \int_0^{\frac{\pi}{2}} \frac{\cos\left(nx\right) - 1}{\sin\left(x\right)}\, dx$ yields:

$$I_2 = \int_0^{\frac{\pi}{2}} \frac{\cos\left(2x\right) - 1}{\sin\left(x\right)}\, dx$$

We have to work this out. From chapter 4 on trigonometry we have the following identity:

(4.54) $$\cos\left(2x\right) = 1 - 2\sin^2\left(x\right)$$

Substituting this into the above integral gives

$$\begin{aligned} I_2 &= \int_0^{\frac{\pi}{2}} \frac{\cos\left(2x\right) - 1}{\sin\left(x\right)}\, dx \\ &= \int_0^{\frac{\pi}{2}} \frac{1 - 2\sin^2\left(x\right) - 1}{\sin\left(x\right)}\, dx \\ &= -2\int_0^{\frac{\pi}{2}} \frac{\sin^{\not 2}\left(x\right)}{\not\sin\left(x\right)}\, dx = -2\int_0^{\frac{\pi}{2}} \sin\left(x\right)\, dx = 2\left[\cos\left(x\right)\right]_0^{\frac{\pi}{2}} = 2\left[0 - 1\right] = -2 \end{aligned}$$

Substituting this $I_2 = -2$ into (**) gives

$$I_4 = I_2 - \frac{2}{3} = -2 - \frac{2}{3} = -\frac{8}{3}$$