# Complete Solutions to Supplementary Exercises on Limits

1. (a) We are asked to show that $\lim\limits_{n\to\infty}\frac{n^n}{(2n)!}=0$. We first show that $\sum\limits_{n=1}^{\infty}\frac{n^n}{(2n)!}$ converges. Using the ratio test with $a_n=\frac{n^n}{(2n)!}$ then $a_{n+1}=\frac{(n+1)^{n+1}}{(2n+2)!}$ and

$$L=\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)=\lim_{n\to\infty}\left(\frac{(n+1)^{n+1}}{(2n+2)!}\frac{(2n)!}{n^n}\right)$$

$$=\lim_{n\to\infty}\left(\frac{\cancel{(2n)}!}{(2n+2)(2n+1)\cancel{(2n)}!}\frac{(n+1)^n(n+1)}{n^n}\right)$$

$$=\lim_{n\to\infty}\left(\frac{n+1}{4n^2+6n+2}\left(1+\frac{1}{n}\right)^n\right)$$

Now $\lim\limits_{n\to\infty}\left(\frac{n+1}{4n^2+6n+2}\right)=0$ and we have the well-known limit

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$$

Substituting these $\lim\limits_{n\to\infty}\left(\frac{n+1}{4n^2+6n+2}\right)=0$ and $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$ into the above gives

$$L=\lim_{n\to\infty}\left(\frac{n+1}{4n^2+6n+2}\left(1+\frac{1}{n}\right)^n\right)=0\times e=0$$

Since $L=0<1$ so the series $\sum\limits_{n=1}^{\infty}\frac{n^n}{(2n)!}$ converges which implies that

$$\lim_{n\to\infty}\left(\frac{n^n}{(2n)!}\right)=0$$

(b) We need to show that $\lim\limits_{n\to\infty}\frac{(2n)!}{a^{n!}}=0$. Consider the infinite series

$$\sum_{n=1}^{\infty}\frac{(2n)!}{a^{n!}}\quad\text{where } a>1$$

Applying the ratio test with $b_n=\frac{(2n)!}{a^{n!}}$ then $b_{n+1}=\frac{(2n+2)!}{a^{(n+1)!}}$ and so

$$L = \lim_{n\to\infty}\left(\frac{(2n+2)!}{a^{(n+1)!}} \div \frac{(2n)!}{a^{n!}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(2n+2)!}{a^{(n+1)!}} \times \frac{a^{n!}}{(2n)!}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(2n+2)(2n+1)\,\cancel{(2n)}!}{a^{(n+1)!-n!}} \times \frac{1}{\cancel{(2n)}!}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(2n+2)(2n+1)}{a^{n!n}}\right) = \lim_{n\to\infty}\left(\frac{4n^2+6n+2}{(a^n)^{n!}}\right)$$

To evaluate this limit we use the *squeeze theorem* which says:

If $x_n \le y_n \le z_n$ for all natural numbers $n$ and $\lim_{n\to\infty}(x_n) = \lim_{n\to\infty}(z_n) = K$ then

$$\lim_{n\to\infty}(y_n) = K$$

For all $n$ we have $(a^n)^{n!} \ge a^n$ so

$$\frac{4n^2+6n+2}{(a^n)^{n!}} \le \frac{4n^2+6n+2}{a^n}$$

Since $n \to \infty$ so $4n^2+6n+2$ is unbounded which implies there are natural numbers $n$ such that $4n^2+6n+2 \ge a$ because $a$ is a real number $>1$. (Choose $n > a$). Therefore

$$\frac{4n^2+6n+2}{(a^n)^{n!}} \ge \frac{a}{(a^n)^{n!}} = \frac{1}{(a^n)^{n!}a^{-1}} = \frac{1}{a^{nn!-1}}$$

Combining these inequalities we have

$$\frac{1}{a^{nn!-1}} \le \frac{4n^2+6n+2}{(a^n)^{n!}} \le \frac{4n^2+6n+2}{a^n}$$

Now $\lim_{n\to\infty}\left(\frac{1}{a^{nn!-1}}\right) = \lim_{n\to\infty}\left(\frac{1}{a}\right)^{nn!-1} = 0$ because $a > 1$. We can apply L'Hopital's rule to find the other limit. This is given by:

*L'Hopital's rule*

*Let* $f(x)$ and $g(x)$ *be differentiable and we have*

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0} \quad \text{or} \quad \lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$$

*Where $a$ is any real number, infinity or minus infinity. Then*

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Applying this rule gives

$$\lim_{n\to\infty}\left(\frac{4n^2+6n+2}{a^n}\right)=\lim_{n\to\infty}\left(\frac{8n+6}{a^n\ln(a)}\right)=\lim_{n\to\infty}\left(\frac{8}{a^n\ln^2(a)}\right)=0 \text{ again because } a>1.$$

Now applying the squeeze theorem we have

$$L=\lim_{n\to\infty}\left|\frac{4n^2+6n+2}{\left(a^n\right)^{n!}}\right|=0<1$$

By the ratio test $\sum_{n=1}^{\infty}\frac{(2n)!}{a^{n!}}$ converges which implies $\lim_{n\to\infty}\frac{(2n)!}{a^{n!}}=0$.

(c) We are asked to show $\lim_{n\to\infty}\frac{a^n}{n!}=0$. Again we consider the series $\sum_{n=1}^{\infty}\frac{a^n}{n!}$. If we can show that this series converges then the limiting value of the summand is equal to zero. Applying the ratio test with $b_n=\frac{a^n}{n!}$ which implies

$b_{n+1}=\frac{a^{n+1}}{(n+1)!}$ .

$$\begin{aligned}L&=\lim_{n\to\infty}\left[\frac{a^{\not n+1}}{(n+1)!}\times\frac{n!}{\not a^{n}}\right]\\&=\lim_{n\to\infty}\left[\frac{a}{(n+1)\not n!}\times\frac{\not n!}{1}\right]\\&=\lim_{n\to\infty}\left[\frac{a}{n+1}\right]=0\end{aligned}$$

Since $L=0<1$ so by the ratio test $\sum_{n=1}^{\infty}\frac{a^n}{n!}$ converges therefore $\lim_{n\to\infty}\frac{a^n}{n!}=0$.

(d) We need to show that $\lim_{n\to\infty}\frac{n^n}{(n!)^2}=0$. Consider the series $\sum_{n=1}^{\infty}\frac{n^n}{(n!)^2}$. Testing this series for convergence by the ratio test with

$$a_n=\frac{n^n}{(n!)^2}\quad\Rightarrow\quad a_{n+1}=\frac{(n+1)^{n+1}}{\left((n+1)!\right)^2}$$

We have

$$L = \lim_{n\to\infty}\left[\frac{(n+1)^{n+1}}{((n+1)!)^2}\times\frac{(n!)^2}{n^n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{(n+1)^n\,\cancel{(n+1)}}{(n+1)^{\cancel{2}}\,\cancel{(n!)^2}}\times\frac{\cancel{(n!)^2}}{n^n}\right] = \lim_{n\to\infty}\left[\frac{1}{n+1}\left(1+\frac{1}{n}\right)^n\right]$$

Since $\lim_{n\to\infty}\left(\frac{1}{n+1}\right)=0$ and $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ so

$$L = \lim_{n\to\infty}\left[\frac{1}{n+1}\left(1+\frac{1}{n}\right)^n\right] = \lim_{n\to\infty}\left(\frac{1}{1+n}\right)\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 0\times e = 0$$

We have $L = 0 < 1$ therefore by the ratio test $\sum_{n=1}^{\infty}\frac{n^n}{(n!)^2}$ converges which

implies $\lim_{n\to\infty}\frac{n^n}{(n!)^2}=0$.

(e) We need to show $\lim_{n\to\infty}\frac{(n!)^n}{n^{n^2}}=0$. Let us test the series $\sum_{n=1}^{\infty}\frac{(n!)^n}{n^{n^2}}$ by the

ratio test with $a_n = \frac{(n!)^n}{n^{n^2}}$ then $a_{n+1} = \frac{((n+1)!)^{n+1}}{(n+1)^{(n+1)^2}}$:

$$L = \lim_{n\to\infty}\left[\frac{((n+1)!)^{n+1}}{(n+1)^{(n+1)^2}}\times\frac{n^{n^2}}{(n!)^n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{(n+1)^{n+1}(n!)^{n+1}}{(n+1)^{n^2+2n+1}}\times\frac{n^{n^2}}{(n!)^n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{\cancel{(n!)^n}\,n!}{(n+1)^{n^2}(n+1)^{2n+1-n-1}}\times\frac{n^{n^2}}{\cancel{(n!)^n}}\right]$$

$$= \lim_{n\to\infty}\left[\frac{n!}{(n+1)^n}\times\left(\frac{n}{n+1}\right)^{n^2}\right]$$

Considering each of the products in the limit separately gives

$$\lim_{n\to\infty}\frac{n!}{(n+1)^n}=\lim_{n\to\infty}\left[\frac{1}{n+1}\times\frac{2}{n+1}\times\cdots\times\frac{n}{n+1}\right]$$

Now $\lim_{n\to\infty}\left[\frac{1}{n+1}\right]=0$ but $\lim_{n\to\infty}\left[\frac{n}{n+1}\right]=1$ therefore

$$\lim_{n\to\infty}\frac{n!}{(n+1)^n}=0\times\cdots\times 1=0$$

For the other term in the product we have

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n^2}=\lim_{n\to\infty}\left(\frac{1}{\frac{n+1}{n}}\right)^{n^2}=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}=\lim_{n\to\infty}\left(\frac{1}{\left(1+\frac{1}{n}\right)^{n}}\right)^{n}=\lim_{n\to\infty}\left(\frac{1}{e}\right)^{n}=0$$

Hence

$$L=\lim_{n\to\infty}\left[\frac{n!}{(n+1)^n}\times\left(\frac{n}{n+1}\right)^{n^2}\right]=0<1$$

By the ratio test $\sum_{n=1}^{\infty}\frac{(n!)^n}{n^{n^2}}$ converges so $\lim_{n\to\infty}\frac{(n!)^n}{n^{n^2}}=0$.

2. (a) We need to expand $\frac{2[\tan(x)-\sin(x)]-x^3}{x^5}$. We have the following general power series:

(7.16) $$\sin(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$$

(7.18) $$\tan(x)=x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\cdots$$

Substituting these into the above given expression yields

$$\frac{2\left[\tan\left(x\right)-\sin\left(x\right)\right]-x^3}{x^5}=\frac{2\left[x+\dfrac{x^3}{3}+\dfrac{2x^5}{15}+\dfrac{17x^7}{315}+\cdots-x+\dfrac{x^3}{6}-\dfrac{x^5}{120}+\dfrac{x^7}{5040}-\cdots\right]-x^3}{x^5}$$

$$=\frac{2\left[\left(x-x\right)+\left(\dfrac{x^3}{3}+\dfrac{x^3}{6}\right)+\left(\dfrac{2x^5}{15}-\dfrac{x^5}{120}\right)+\left(\dfrac{17x^7}{315}+\dfrac{x^7}{5040}\right)+\cdots\right]-x^3}{x^5}$$

$$=\frac{2\left[\dfrac{x^3}{2}+\dfrac{15x^5}{120}+\dfrac{13}{240}x^7+\cdots\right]-x^3}{x^5}$$

$$=\frac{\dfrac{30x^5}{120}+\dfrac{13}{120}x^7+\cdots}{x^5}=\frac{1}{4}+\frac{13}{120}x^2+\cdots$$

Now evaluating the limit gives

$$\lim_{x\to 0}\frac{2\left[\tan\left(x\right)-\sin\left(x\right)\right]-x^3}{x^5}=\lim_{x\to 0}\left(\frac{1}{4}+\frac{13}{120}x^2+\cdots\right)=\frac{1}{4}$$

(b) We need to evaluate $\lim\limits_{x\to 0}\dfrac{\ln\left(1+x+x^2\right)+\ln\left(1-x+x^2\right)}{x\left(e^x-1\right)}$. Applying the rules of logs; $\ln\left(A\right)+\ln\left(B\right)=\ln\left(AB\right)$ to the numerator gives

$$\begin{aligned}\ln\left(1+x+x^2\right)+\ln\left(1-x+x^2\right)&=\ln\left[\left(1+x+x^2\right)\left(1-x+x^2\right)\right]\\&=\ln\left(1-x+x^2+x-x^2+x^3+x^2-x^3+x^4\right)\\&=\ln\left(1+x^2+x^4\right)\end{aligned}$$

Using the power series:

(7.21) $$\ln\left(1+x\right)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots \text{ provided } -1<x\le 1$$

On the above gives

$$\begin{aligned}\ln\left(1+x^2+x^4\right)&=\left(x^2+x^4\right)-\frac{1}{2}\left(x^2+x^4\right)^2+\cdots\\&=\left(x^2+x^4\right)-\frac{x^4}{2}\left(1+x^2\right)^2+\cdots\qquad(*)\end{aligned}$$

Now using the power series

(7.15) $$e^x=1+x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots$$

On the denominator $x\left(e^x-1\right)$ gives

$$\begin{aligned}x\left(e^x-1\right)&=x\left(1+x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots-1\right)\\&=x\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots\right)=x^2\left(1+\frac{x}{2}+\frac{x^2}{3}+\cdots\right)\qquad(**)\end{aligned}$$

Substituting these (*) and (**) into the given limit yields

$$\lim_{x\to 0}\frac{\ln\left(1+x+x^2\right)+\ln\left(1-x+x^2\right)}{x\left(e^x-1\right)}=\lim_{x\to 0}\frac{\left(x^2+x^4\right)-\frac{x^4}{2}\left(1+x^2\right)^2+\cdots}{x^2\left(1+\frac{x}{2}+\frac{x^2}{3}+\cdots\right)}$$

$$=\lim_{x\to 0}\frac{1+x^2-\frac{x^2}{2}\left(1+x^2\right)^2+\cdots}{1+\frac{x}{2}+\frac{x^2}{3}+\cdots}\quad\left[\text{Cancelling } x^2\right]$$

$$=1$$

(c) We need to evaluate $\lim_{x\to\infty}\left\{x-x^2\ln\left(1+\frac{1}{x}\right)\right\}$. Using

(7.21) $$\ln\left(1+x\right)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots \text{ provided } -1<x\le 1$$

We have

$$x-x^2\ln\left(1+\frac{1}{x}\right)=x-x^2\left[\frac{1}{x}-\frac{1}{2x^2}+\frac{1}{3x^3}-\frac{1}{4x^4}+\cdots\right]$$

$$=x-x+\frac{1}{2}-\frac{1}{3x}+\frac{1}{4x^2}+\cdots$$

$$=\frac{1}{2}-\frac{1}{3x}+\frac{1}{4x^2}+\cdots$$

Working out the limit gives

$$\lim_{x\to\infty}\left\{x-x^2\ln\left(1+\frac{1}{x}\right)\right\}=\lim_{x\to\infty}\left\{\frac{1}{2}-\frac{1}{3x}+\frac{1}{4x^2}+\cdots\right\}=\frac{1}{2}$$

(d) We are given $\lim_{x\to 0}\left\{\frac{1}{x^2}-\cot^2\left(x\right)\right\}$. Recall $\cot^2\left(x\right)=\frac{1}{\tan^2\left(x\right)}$.

Rearranging the expression under the limit sign:

$$\frac{1}{x^2}-\cot^2\left(x\right)=\frac{1}{x^2}-\frac{1}{\tan^2\left(x\right)}=\frac{\tan^2\left(x\right)-x^2}{x^2\tan^2\left(x\right)}$$

Applying

(7.18) $$\tan\left(x\right)=x+\frac{x^3}{3}+\frac{2x^5}{15}+\cdots$$

This to $\tan^2\left(x\right)$ gives

$$\tan^2\left(x\right)=\left(x+\frac{x^3}{3}+\cdots\right)^2$$

$$=x^2+\frac{2x^4}{3}+\frac{x^6}{9}+\cdots$$

Therefore we have

$$\frac{\tan^2(x)-x^2}{x^2\tan^2(x)}=\frac{x^2+\frac{2x^4}{3}+\frac{x^6}{9}+\cdots-x^2}{x^2\left[x^2+\frac{2x^4}{3}+\frac{x^6}{9}+\cdots\right]}=\frac{\cancel{x^4}\left[\frac{2}{3}+\frac{x^2}{9}+\cdots\right]}{\cancel{x^4}\left[1+\frac{2x^2}{3}+\frac{x^4}{9}+\cdots\right]}$$

Sticking in the limit yields

$$\lim_{x\to 0}\left[\frac{\tan^2(x)-x^2}{x^2\tan^2(x)}\right]=\lim_{x\to 0}\frac{\frac{2}{3}+\frac{x^2}{9}+\cdots}{1+\frac{2x^2}{3}+\frac{x^4}{9}+\cdots}=\frac{2}{3}$$

Hence $\lim_{x\to 0}\left\{\frac{1}{x^2}-\cot^2(x)\right\}=\frac{2}{3}$.

(e) We need to evaluate the limit, $\lim_{x\to 0}\left\{\frac{1}{x^2}-\frac{\cot(x)}{x}\right\}$. We have

$$\frac{1}{x^2}-\frac{\cot(x)}{x}=\frac{1}{x^2}-\frac{1}{x\tan(x)}=\frac{x\tan(x)-x^2}{x^3\tan(x)}$$

Using the power series for tan:

$$\tan(x)=x+\frac{x^3}{3}+\frac{2x^5}{15}+\cdots \tag{7.18}$$

Gives

$$\frac{x\tan(x)-x^2}{x^3\tan(x)}=\frac{x\left[x+\frac{x^3}{3}+\frac{2x^5}{15}+\cdots\right]-x^2}{x^3\left[x+\frac{x^3}{3}+\frac{2x^5}{15}+\cdots\right]}$$

$$=\frac{\frac{x^4}{3}+\frac{2x^6}{15}+\cdots}{x^4\left[1+\frac{x^2}{3}+\frac{2x^4}{15}+\cdots\right]}=\frac{\cancel{x^4}\left[\frac{1}{3}+\frac{2x^2}{15}+\cdots\right]}{\cancel{x^4}\left[1+\frac{x^2}{3}+\frac{2x^4}{15}+\cdots\right]}=\frac{\frac{1}{3}+\frac{2x^2}{15}+\cdots}{1+\frac{x^2}{3}+\frac{2x^4}{15}+\cdots}$$

Now evaluating the limit

$$\lim_{x\to 0}\left\{\frac{1}{x^2}-\frac{\cot(x)}{x}\right\}=\lim_{x\to 0}\left\{\frac{\frac{1}{3}+\frac{2x^2}{15}+\cdots}{1+\frac{x^2}{3}+\frac{2x^4}{15}+\cdots}\right\}=\frac{1}{3}$$

(f) We are asked to evaluate $\lim_{x\to 0}\left\{\frac{2+\cos(x)}{x^3\sin(x)}-\frac{3}{x^4}\right\}$.

Cross multiplying the expression under the limit sign and taking out a common factor gives:

$$\lim_{x\to 0}\left\{\frac{2+\cos\left(x\right)}{x^3\sin\left(x\right)}-\frac{3}{x^4}\right\}=\lim_{x\to 0}\left\{\frac{1}{x^3}\left[\frac{2+\cos\left(x\right)}{\sin\left(x\right)}-\frac{3}{x}\right]\right\}$$

$$=\lim_{x\to 0}\left\{\frac{1}{x^3}\left[\frac{2x+x\cos\left(x\right)-3\sin\left(x\right)}{x\sin\left(x\right)}\right]\right\}$$

$$=\lim_{x\to 0}\left\{\frac{1}{x^4}\left[\frac{2x+x\cos\left(x\right)-3\sin\left(x\right)}{\sin\left(x\right)}\right]\right\}$$

Using the sin and cosine power series:

(7.16) $$\sin\left(x\right)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots$$

(7.17) $$\cos\left(x\right)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots$$

In the expression in the square brackets:

$$\frac{2x+x\cos\left(x\right)-3\sin\left(x\right)}{x\sin\left(x\right)}=\frac{2x+x\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots\right)-3\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots\right)}{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots}$$

$$=\frac{\left(3x-3x\right)+\left(-\frac{x^3}{2}+\frac{x^3}{2}\right)+\left(\frac{x^5}{24}-\frac{x^5}{40}\right)-\left(\frac{x^7}{720}-\frac{3x^7}{5040}\right)+\cdots}{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots}$$

$$=\frac{\frac{x^5}{60}-\frac{x^7}{1260}-\dots}{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots}=\frac{\frac{x^4}{60}-\frac{x^6}{1260}-\dots}{1-\frac{x^2}{3!}+\frac{x^4}{5!}-\dots}\quad\left[\text{Cancelling } x\text{'s}\right]$$

Evaluating the given limit:

$$\lim_{x\to 0}\left\{\frac{2+\cos\left(x\right)}{x^3\sin\left(x\right)}-\frac{3}{x^4}\right\}=\lim_{x\to 0}\left\{\frac{1}{x^4}\left[\frac{2x+x\cos\left(x\right)-3\sin\left(x\right)}{\sin\left(x\right)}\right]\right\}$$

$$=\lim_{x\to 0}\left\{\frac{1}{x^4}\left[\frac{\frac{x^4}{60}-\frac{x^6}{1260}-\dots}{1-\frac{x^2}{3!}+\frac{x^4}{5!}-\dots}\right]\right\}\quad\left[\text{From above}\right]$$

$$=\lim_{x\to 0}\left\{\left[\frac{\frac{1}{60}-\frac{x^2}{1260}-\dots}{1-\frac{x^2}{3!}+\frac{x^4}{5!}-\dots}\right]\right\}=\frac{1}{60}$$

(g) First we need to find the Taylor expansion of $\dfrac{x+\ln\left(\sqrt{1+x^2}-x\right)}{x^3}$.

Considering the ln term first, we have

$$\begin{aligned}\ln\left(\sqrt{1+x^2}-x\right) &= \ln\left(\sqrt{1+x^2}-x\right)-\ln\left(\sqrt{1+x^2}\right)+\ln\left(\sqrt{1+x^2}\right)\\ &= \ln\left(\frac{\sqrt{1+x^2}-x}{\sqrt{1+x^2}}\right)+\ln\left[\left(1+x^2\right)^{1/2}\right] \qquad \left[\begin{array}{l}\text{Using}\\ \ln(A)-\ln(B)=\ln\left(\dfrac{A}{B}\right)\end{array}\right]\\ &= \ln\left(1-\frac{x}{\sqrt{1+x^2}}\right)+\frac{1}{2}\ln\left(1+x^2\right)\end{aligned}$$

Applying the power series

(7.21) $$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots \text{ provided } -1<x\le 1$$

To the above expression gives

$$\begin{aligned}\ln\left(1-\frac{x}{\sqrt{1+x^2}}\right)+\frac{1}{2}\ln\left(1+x^2\right) &= -\frac{x}{\sqrt{1+x^2}}-\frac{1}{2}\frac{x^2}{1+x^2}-\frac{1}{3}\frac{x^3}{\left(1+x^2\right)^{3/2}}-\frac{1}{4}\frac{x^4}{\left(1+x^2\right)^2}+\cdots\\ &\qquad +\frac{1}{2}\left[x^2-\frac{x^4}{2}+\cdots\right]\\ &= -\frac{x}{\sqrt{1+x^2}}-\frac{x^2}{2}\left[\frac{1}{1+x^2}-1\right]-\frac{1}{3}\frac{x^3}{\left(1+x^2\right)^{3/2}}-\frac{x^4}{4}\left[\frac{1}{\left(1+x^2\right)^2}+1\right]+\cdots\end{aligned}$$

Substituting this into the given expression $\dfrac{x+\ln\left(\sqrt{1+x^2}-x\right)}{x^3}$ yields

$$\begin{aligned}&\frac{x-\dfrac{x}{\sqrt{1+x^2}}-\dfrac{x^2}{2}\left[\dfrac{1}{1+x^2}-1\right]-\dfrac{1}{3}\dfrac{x^3}{\left(1+x^2\right)^{3/2}}-\dfrac{x^4}{4}\left[\dfrac{1}{\left(1+x^2\right)^2}+1\right]+\cdots}{x^3}\\ &= \frac{x\left[1-\dfrac{1}{\sqrt{1+x^2}}\right]-\dfrac{x^2}{2}\left[\dfrac{-x^2}{1+x^2}\right]-\dfrac{1}{3}\dfrac{x^3}{\left(1+x^2\right)^{3/2}}-\dfrac{x^4}{4}\left[\dfrac{1}{\left(1+x^2\right)^2}+1\right]+\cdots}{x^3}\\ &= \frac{1}{x^2}\left[1-\frac{1}{\sqrt{1+x^2}}\right]+\frac{x}{2}\left[\frac{1}{1+x^2}\right]-\frac{1}{3}\frac{1}{\left(1+x^2\right)^{3/2}}-\frac{x}{4}\left[\frac{1}{\left(1+x^2\right)^2}+1\right]+\cdots \qquad (\dagger)\end{aligned}$$

The first term we can expand by using the binomial theorem:

$$\frac{1}{x^2}\left[1-\frac{1}{\sqrt{1+x^2}}\right]=\frac{1}{x^2}\left[1-\left(1+x^2\right)^{-1/2}\right]$$
$$=\frac{1}{x^2}\left[1-\left(1-\frac{1}{2}x^2+\frac{\left(-1/2\right)\left(-3/2\right)}{2!}x^4-\cdots\right)\right]$$
$$=\frac{1}{x^2}\left[\frac{1}{2}x^2-\frac{3}{8}x^4+\cdots\right]=\frac{1}{2}-\frac{3}{8}x^2+\cdots$$

Evaluating the limit for this and the remaining parts gives

$$\lim_{x\to 0}\frac{1}{x^2}\left[1-\frac{1}{\sqrt{1+x^2}}\right]=\lim_{x\to 0}\left(\frac{1}{2}-\frac{3}{8}x^2+\cdots\right)=\frac{1}{2}$$

$$\lim_{x\to 0}\left(\frac{x}{2}\left[\frac{1}{1+x^2}\right]-\frac{1}{3}\frac{1}{\left(1+x^2\right)^{3/2}}-\frac{x}{4}\left[\frac{1}{\left(1+x^2\right)^2}+1\right]+\cdots\right)=0-\frac{1}{3}-0+0\cdots$$

Putting all this together in $\left(\dagger\right)$ gives

$$\lim_{x\to 0}\left[\frac{x+\ln\left(\sqrt{1+x^2}-x\right)}{x^3}\right]=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$$

3. We are asked to expand $f\left(x\right)=\dfrac{1+x}{\left(1-x\right)^3}$. Differentiating this term gives

$$f'\left(x\right)=\frac{\left(1-x\right)^3+3\left(1+x\right)\left(1-x\right)^2}{\left(1-x\right)^6}$$
$$\underbrace{=}_{\text{Cancelling }\left(1-x\right)^2}\frac{1-x+3\left(1+x\right)}{\left(1-x\right)^4}=\frac{4+2x}{\left(1-x\right)^4}$$

Repeatedly differentiating gives

$$f''\left(x\right)=\left[\frac{4+2x}{\left(1-x\right)^4}\right]'=\frac{2\left(1-x\right)^4+4\left(4+2x\right)\left(1-x\right)^3}{\left(1-x\right)^8}$$
$$\underbrace{=}_{\text{Cancelling }\left(1-x\right)^3}\frac{2-2x+16+8x}{\left(1-x\right)^5}=\frac{18+6x}{\left(1-x\right)^5}$$

$$f'''(x) = \left[\frac{18+6x}{(1-x)^5}\right]' = \frac{6(1-x)^5 + 5(18+6x)(1-x)^4}{(1-x)^{10}}$$

$$\underbrace{=}_{\text{Cancelling } (1-x)^4} \frac{6-6x+90+30x}{(1-x)^6} = \frac{96+24x}{(1-x)^6}$$

$$f^{iv}(x) = \left[\frac{96+24x}{(1-x)^6}\right]' = \frac{24(1-x)^6 + 6(96+24x)(1-x)^5}{(1-x)^{12}}$$

$$\underbrace{=}_{\text{Cancelling } (1-x)^5} \frac{24-24x+576+144x}{(1-x)^7} = \frac{600+120x}{(1-x)^7}$$

Substituting $x = 0$ into these gives

$$f(0) = \frac{1+0}{(1-0)^3} = 1$$

$$f'(0) = \frac{4+2(0)}{(1-0)^4} = 4$$

$$f''(0) = \frac{18+6(0)}{(1-0)^5} = 18$$

$$f'''(0) = \frac{96+24(0)}{(1-0)^6} = 96$$

$$f^{iv}(0) = \frac{600+120(0)}{(1-0)^7} = 600$$

Putting these into the general Macluarin series formula (because the neighbourhood is zero) gives

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{iv}(0)}{4!}x^4 + \cdots \\ &= 1 + 4x + \frac{18}{2}x^2 + \frac{96}{6}x^3 + \frac{600}{24}x^4 + \cdots \\ &= 1 + 4x + 9x^2 + 16x^3 + 25x^4 + \cdots \qquad (*) \end{aligned}$$

We are also asked to find the sum of the series $\sum_{m=1}^{\infty} \frac{m^2}{2^{m-1}}$:

$$\sum_{m=1}^{\infty} \frac{m^2}{2^{m-1}} = 1 + 4\left(\frac{1}{2}\right) + 9\left(\frac{1}{2}\right)^2 + 16\left(\frac{1}{2}\right)^3 + 25\left(\frac{1}{2}\right)^4 + 36\left(\frac{1}{2}\right)^5 + \ldots$$

Putting $x = \frac{1}{2}$ into the above expression we evaluated in (*) gives

$$f\left(\frac{1}{2}\right) = 1 + 4\left(\frac{1}{2}\right) + 9\left(\frac{1}{2}\right)^2 + 16\left(\frac{1}{2}\right)^3 + 25\left(\frac{1}{2}\right)^4 + 36\left(\frac{1}{2}\right)^5 + \ldots = \sum_{m=1}^{\infty} \frac{m^2}{2^{m-1}}$$

We are given $f\left(x\right) = \frac{1+x}{\left(1-x\right)^3}$ therefore

$$\sum_{m=1}^{\infty} \frac{m^2}{2^{m-1}} = f\left(\frac{1}{2}\right) = \frac{1+\frac{1}{2}}{\left(1-\frac{1}{2}\right)^3} = \frac{3/2}{1/8} = \frac{3}{2} \times \frac{8}{1} = 12$$