SECTION D **Odd and Even Functions**

By the end of this section you will be able to

- understand the meaning of the constant term, A_0, in the Fourier series
- recognise even and odd functions
- obtain Fourier series for even and odd functions

D1 **Constant Term**

Fig. 22 below shows various periodic functions with the value of their constant term A_0 in the Fourier series.

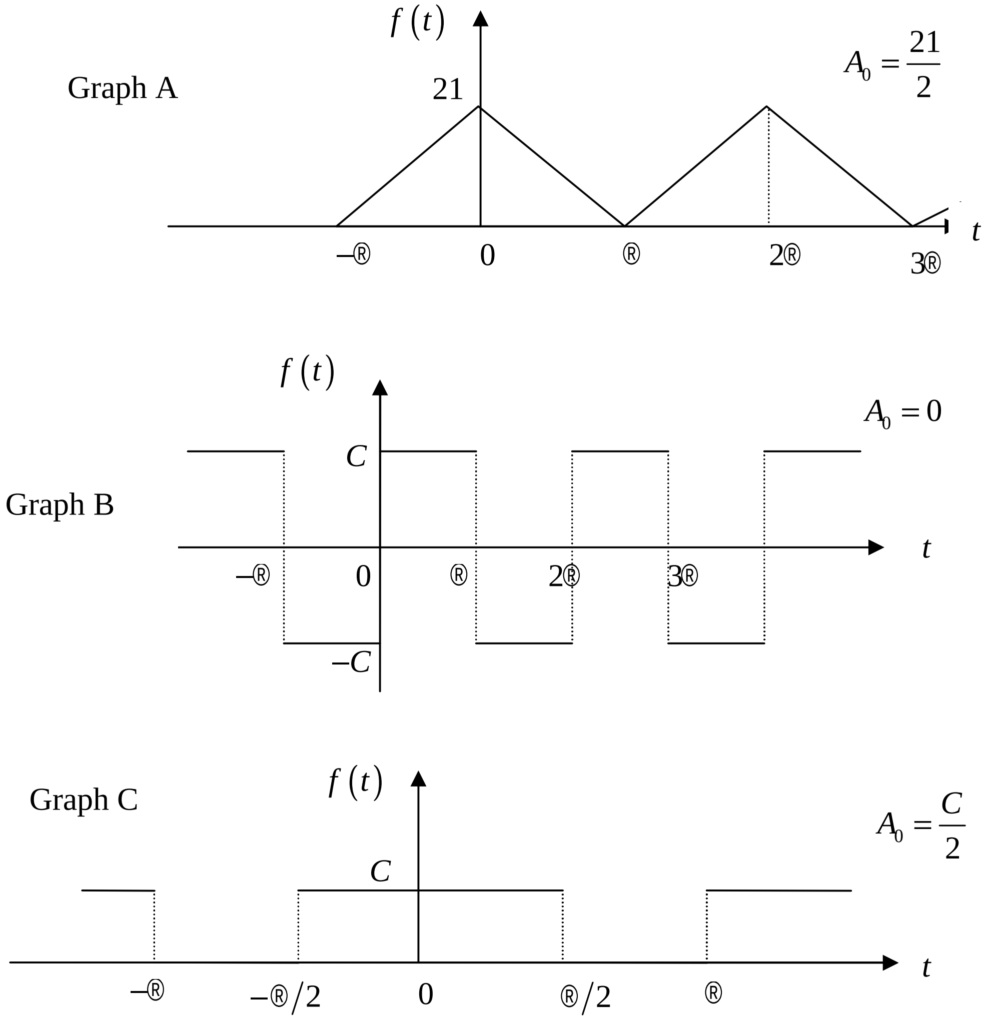

Figure 22

What do you notice about the relationship between the function $f(t)$ *and the constant term* A_0 *?*

A_0 is the *average value* of the function $f(t)$ over one complete period. Remember the formula for the constant term A_0 is given by

(17.3) $$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t$$

What does the integral, $\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t$*, mean?*

If the given function is positive or zero over this period ($-\pi$ to π), that is $f(t) \geq 0$, then $\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t$ is the area under the curve $f(t)$ and the t axis between $-\pi$ to π.

Also if the given function is negative or zero over this period, that is $f(t) \leq 0$, then $\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t$ is the area under the curve $f(t)$ and the t axis between $-\pi$ to π.

Hence if either of these conditions is satisfied then we can rewrite formula (17.3) as

(17.6) $$A_0 = \frac{\text{Area under } f(t) \text{ between } -\pi \text{ to } \pi}{2\pi}$$

As stated above and in section B this A_0 is the average value of the function over a complete period.

If $f(t)$ is a linear function then we can evaluate the constant term A_0 (average value) of the Fourier series without using integration as the following examples reveal.

Example 7 (Signal Processing)

A signal has the following voltage waveform:

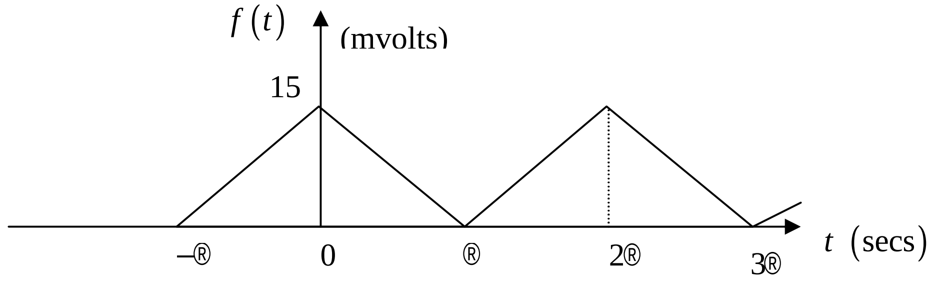

Figure 23

Determine the constant term A_0 of the Fourier series for $f(t)$ shown in Fig. 23.

<u>Solution</u>

Since $f(t)$ is positive or zero between $-\pi$ and π so

(17.6) $$A_0 = \frac{\text{Area under } f(t) \text{ between } -\pi \text{ to } \pi}{2\pi}$$

What is the area under $f(t)$ *between* $-\pi$ *and* π *?*

Area of triangle with width 2π and height 15. Therefore

$$\text{Area} = \frac{1}{\not{2}}\left(15 \times \not{2}\pi\right) = 15\pi \qquad \left[\text{Cancelling 2's}\right]$$

Substituting $\text{Area} = 15\pi$ into formula (17.6) gives

$$A_0 = \frac{15\not{\pi}}{2\not{\pi}} = \frac{15}{2} \qquad \left[\text{Cancelling } \pi\text{'s}\right]$$

Hence the average voltage $A_0 = \dfrac{15}{2}$ mV . You should be able to predict the average value of this function from the outset. (If you applied the integration formula (17.3) then you would get the same answer $A_0 = 15/2$.)

Example 8

Determine the constant term A_0 of the Fourier series of the following function:

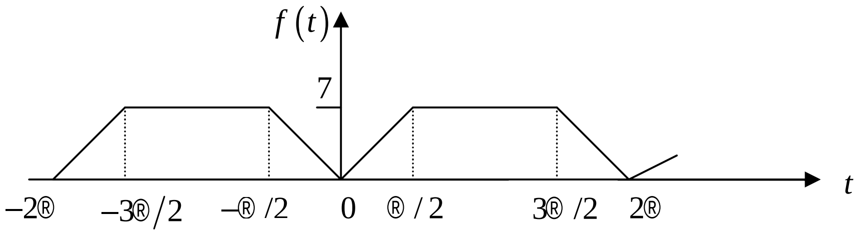

Figure 24

Solution

We consider the function between 0 and 2π because the area under $f(t)$ between -2π and 0 is the same as the area under $f(t)$ between 0 and 2π:

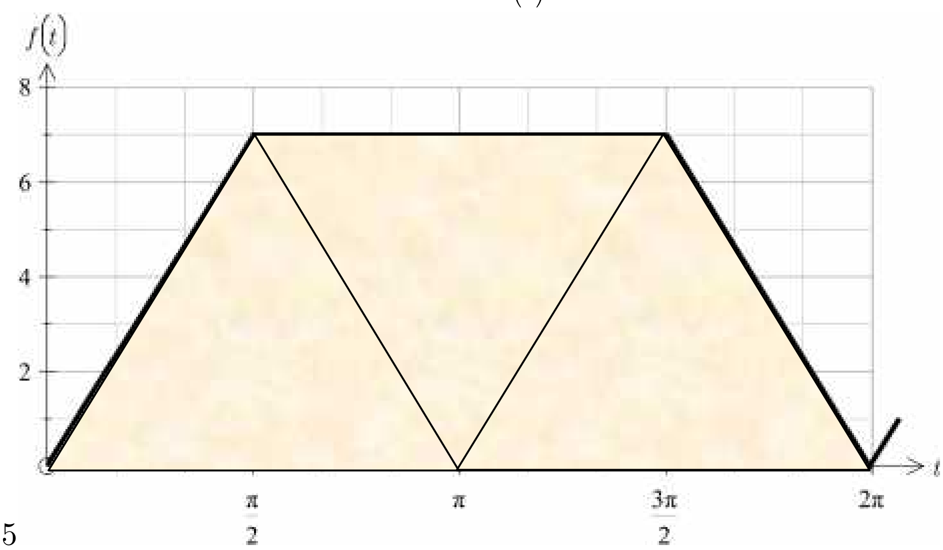

Figure 25

Again the function $f(t)$ is positive or zero over the period 0 to 2π so the average value A_0 is given by

$$A_0 = \frac{\text{Shaded Area}}{2\pi} \qquad (*)$$

How do we find the shaded area?

It is the sum of three triangles of base π and height 7:

$$\text{Shaded area} = 3 \times \frac{(\pi \times 7)}{2} = \frac{21\pi}{2}$$

Substituting this into (*) gives

$$A_0 = \frac{\text{Shaded Area}}{2\pi} = \frac{21\pi/2}{2\pi}$$
$$= \frac{21\not{\pi}}{4\not{\pi}} = \frac{21}{4} \qquad \text{[Cancelling the } \pi\text{'s]}$$

Hence the constant term, A_0, is 21/4.

Recall the constant term A_0 in the Fourier series is the average value of the given function over a complete period.

What is the value of A_0 for

$$f(t) = \begin{cases} 1 & 0 < t < \pi \\ -1 & \pi < t < 2\pi \end{cases}$$

where $f(t)$ has a period 2π ?

Clearly the average value of $f(t)$ is zero. Therefore $A_0 = 0$.

You can see from the above examples that evaluating the constant term A_0 can be straightforward. It is easier to examine the given function $f(t)$ and then decide whether you need to use integration or whether you can evaluate A_0 *without* integration.

D2 **Even Functions**

The following are *all* examples of even functions:

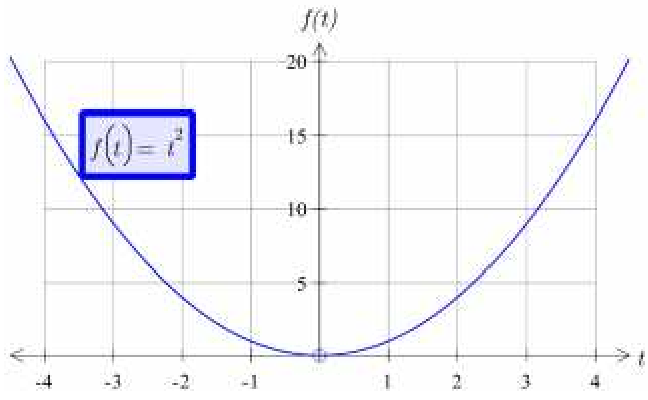

Figure 27(a)

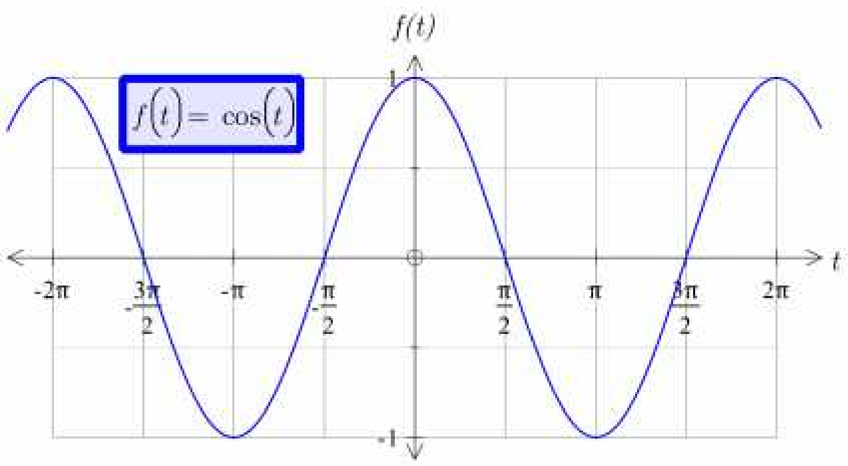

Figure 27(b)

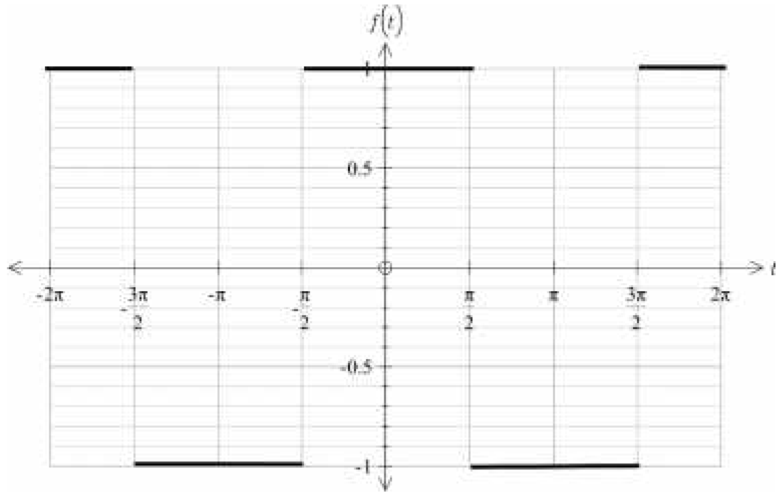

Figure 27(c)

What similarities do all these graphs have?

They are *all* symmetrical about the vertical axis. We can write this in mathematical notation as:

$$f(-t) = f(t)$$

If

(17.7) $f(-t) = f(t)$

then f is an **even** function. This means that $f(\text{Negative } t) = f(\text{Positive } t)$.

Consider the function $f(t) = t^2$ of Fig 27(a). Let $t = 2$ then

$$f(2) = 4$$
$$f(-2) = 4 = f(2)$$

Hence $f(-2) = f(2)$. Consider the function $f(t)$ of Fig 27(c). *What is* $f(-\pi)$ *equal to in terms of* $f(\pi)$*?*

By using the graph of Fig 27(c) we can see that

$$f(-\pi) = -1 \text{ and } f(\pi) = -1$$
$$f(-\pi) = -1 = f(\pi)$$

An even function is symmetrical about the vertical axis.

You are asked to show the following result in the exercises:

$$\cos(t),\ \cos(2t),\ \cos(3t),\ \cdots,\ \cos(kt) \text{ are } \textit{all even functions}$$

D3 **Odd Functions**

The following are *all* examples of odd functions:

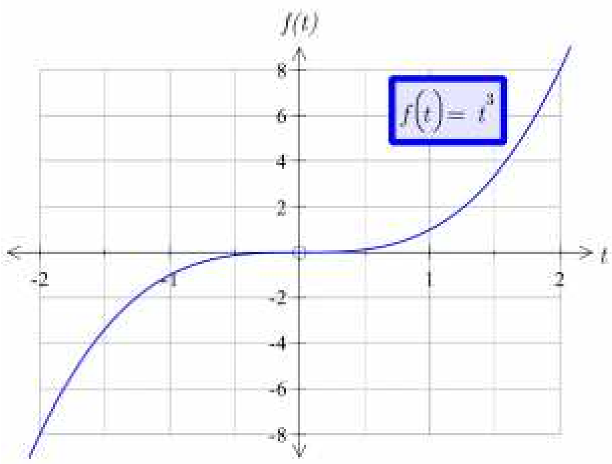

Figure 28(a)

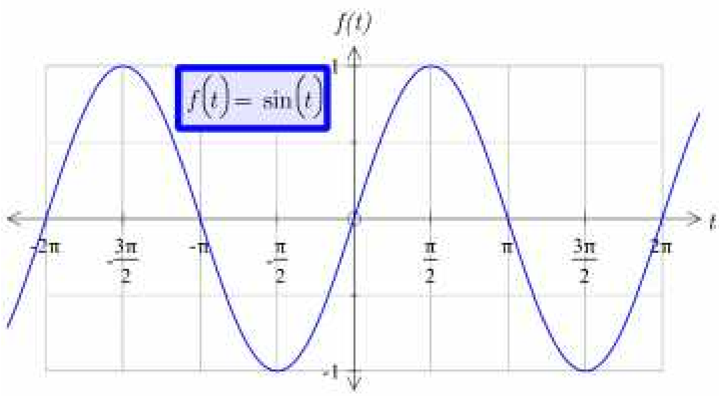

Figure 28(b)

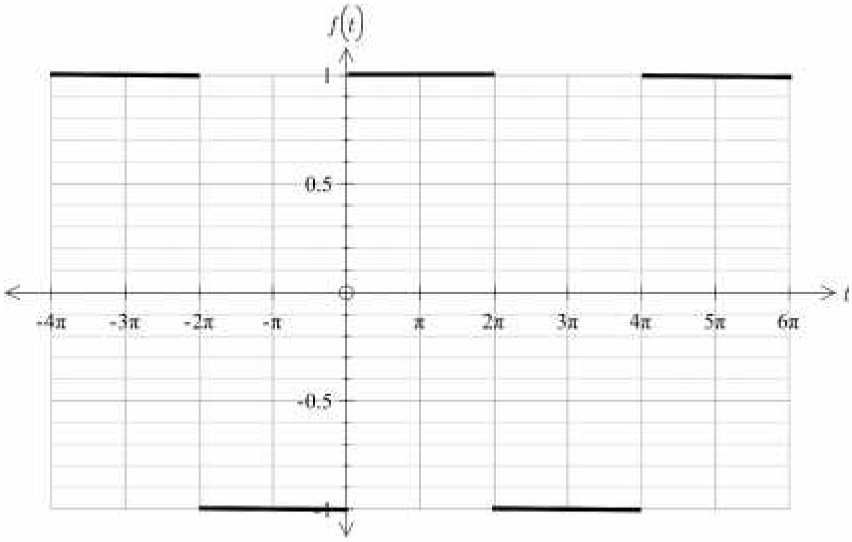

Figure 28(c)

What similarities do all these graphs have?

All these graphs are symmetrical about the origin. That is

$$f(-t) = -f(t)$$

A function is symmetrical about the origin if the graph of the function for negative t is the graph of the function for positive t rotated by 180 degrees. Figure 28 clearly has this property, as can be seen below.

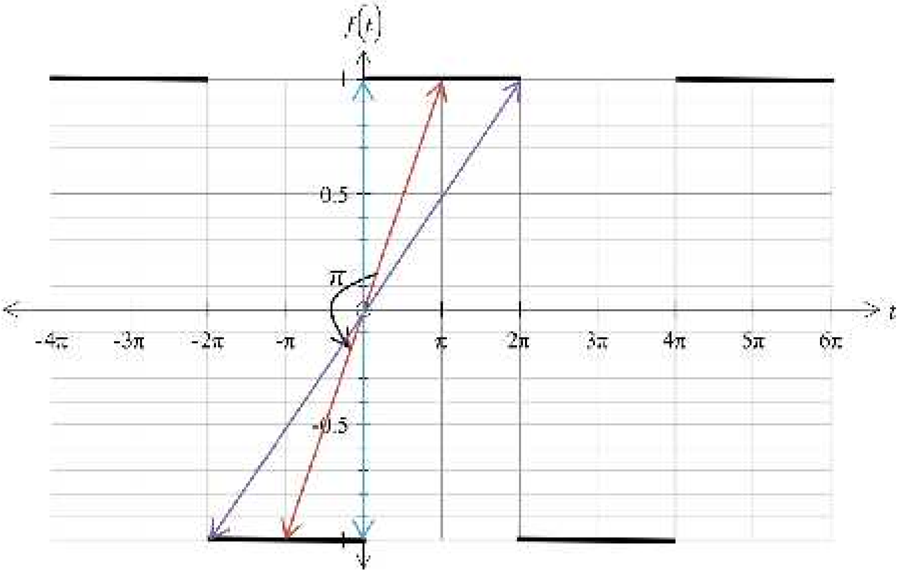

Figure 28(d)

Here, each coloured arrow joins corresponding points. We can see that we can get from each pair of points by a rotation of 180 degrees about the origin. More formally, we give the following definition of an odd function:

> A function $f(t)$ is **odd** if
>
> (17.8) $f(-t) = -f(t)$ $[f(\text{Negative } t) = -f(\text{Positive } t)]$

Consider $f(t) = t^3$ of Fig 28(a). Let $t = 2$ then

$$f(2) = 8 \text{ but } f(-2) = -8$$
$$f(-2) = -8 = -f(2)$$

What is $f(-\pi)$ equal to in terms of $f(\pi)$ for the function of Fig 28(c)?

$$f(-\pi) = -1 \quad \text{but} \quad f(\pi) = 1$$
$$f(-\pi) = -1 = -f(\pi)$$

An odd function is symmetrical about the origin

You are asked to show the following result in the exercises:

$$\sin(t),\ \sin(2t),\ \sin(3t),\ \cdots,\ \sin(kt) \text{ are } \textit{all odd functions}$$

Note that cosine is an *even* function and sine is an *odd* function.

D4 **Odd and Even Functions**

Consider an arbitrary *odd* function:

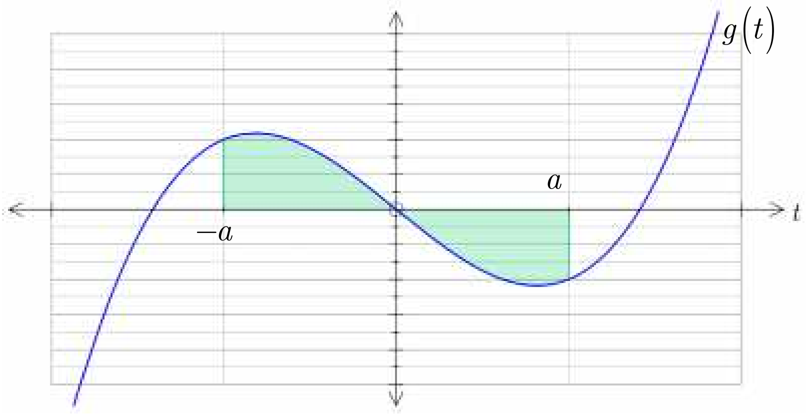

Figure 29

What is $\int_{-a}^{a} g(t)\,\mathrm{d}t$ *equal to?*

By examining the areas we conclude that

$$\int_{-a}^{a} g(t)\,\mathrm{d}t = 0$$

For an odd function we have the following result:

(17.9) $$\int_{-a}^{a} [\text{odd function}]\,\mathrm{d}t = 0$$

For example $\int_{-1}^{1} x^3\,\mathrm{d}x = 0$ because x^3 is an odd function.

Consider an arbitrary *even* function $h(t)$:

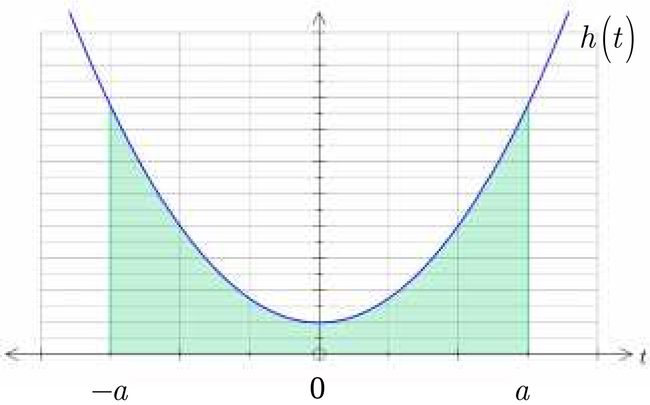

Figure 30

What is $\int_{-a}^{a} h(t)\,\mathrm{d}t$ *equal to in terms of* $\int_{0}^{a} h(t)\,\mathrm{d}t$ *?*

Looking at the areas gives

$$\int_{-a}^{a} h(t)\,\mathrm{d}t = 2\int_{0}^{a} h(t)\,\mathrm{d}t$$

For an even function we have

(17.10) $$\int_{-a}^{a}\left[\text{Even Function}\right]\,\mathrm{d}t = 2\int_{0}^{a}\left[\text{Even Function}\right]\,\mathrm{d}t$$

For example

$$\int_{-1}^{1} x^2\,\mathrm{d}x = 2\int_{0}^{1} x^2\,\mathrm{d}x \text{ because } x^2 \text{ is an even function}$$

Normally it is easier to integrate between 0 to a rather than $-a$ to a.

Both these results, (17.9) and (17.10), have implications for obtaining the Fourier series of odd and even functions. They make life a lot easier for finding the Fourier coefficients provided you have an odd or even function.

D5 **Fourier Series of Odd and Even Functions**

Next we state some results which connect the Fourier series coefficients of an odd and even function. You are asked to derive these results in Exercise 17(d).

For an *odd* function, $f(t)$, with period 2π the Fourier series coefficients are given by

(17.11) $A_0 = 0$ [Average value of $f(t)$]

(17.12) $A_k = 0$ [2 ×(Average value of $f(t)\cos(kt)$)]

(17.13) $B_k = \dfrac{2}{\pi}\displaystyle\int_{0}^{\pi} f(t)\sin(kt)\,\mathrm{d}t$ [2 ×(Average value of $f(t)\sin(kt)$)]

Note there are *no* cosine terms and *no* constant term in the Fourier series of an odd function.

To understand why $A_k = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(t)\cos(kt)\,\mathrm{d}t = 0$ we need to use the following properties of odd and even functions:

$$(\text{odd function})\times(\text{even function}) = \text{odd function} \qquad (*)$$

If $f(t)$ is odd and we know from above that $\cos(kt)$ is even then by (*) we have

$$f(t)\times\cos(kt) = (\text{odd function})\times(\text{even function}) = \text{odd function}$$

By (17.9) $\displaystyle\int_{-\pi}^{\pi}\left[\text{odd function}\right]\mathrm{d}t = 0$ so $A_k = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}\underbrace{f(t)\cos(kt)}_{\text{odd function}}\,\mathrm{d}t = 0$.

Similarly for an *even* function, $f(t)$, the Fourier series coefficients are given by:

(17.14) $A_0 = \dfrac{1}{\pi}\displaystyle\int_{0}^{\pi} f(t)\,\mathrm{d}t$ [Average value of $f(t)$]

(17.15) $$A_k = \frac{2}{\pi}\int_0^{\pi} f(t)\cos(kt)\,\mathrm{d}t \qquad [2 \times (\text{Average value of } f(t)\cos(kt))]$$

(17.16) $$B_k = 0 \qquad [2 \times (\text{Average value of } f(t)\sin(kt))]$$

Note there are *no* sine terms in the Fourier series of an even function.

Similarly $B_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,\mathrm{d}t = 0$ because the given $f(t)$ is an even function and $\sin(kt)$ is an odd function and from above (*) we have:

$$f(t)\times\sin(kt) = (\text{even function})\times(\text{odd function}) = \text{odd function}$$

By (17.9) $\int_{-\pi}^{\pi}[\text{odd function}]\,\mathrm{d}t = 0$ so $B_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(t)\sin(kt)}_{\text{odd function}}\,\mathrm{d}t = 0$.

We will use these results to obtain a Fourier series of an odd or even function. If we can establish that the given function is odd or even then evaluating the Fourier series is not so lengthy as the following example demonstrates.

Example 9 (Electronics)

A sinusoidal waveform is applied to a relay in a circuit. The output response is the waveform $f(\omega t)$ shown below.

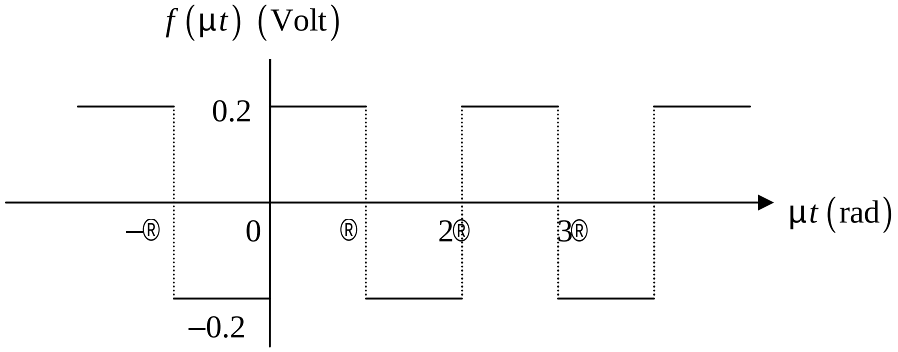

Figure 31

Obtain the Fourier series of $f(\omega t)$.

Solution

What type of function is $f(\omega t)$?

Clearly it is an *odd* function because it is symmetrical about the origin,

$$f(-\omega t) = -f(\omega t)$$

What can we say about the Fourier coefficients?

Since we have an odd function, we conclude that there are *no* cosine or constant terms in the Fourier series. We only need to find the sine coefficients, B_k. We use the above given formula

(17.13) $$B_k = \frac{2}{\pi}\int_0^{\pi} f(t)\sin(kt)\,\mathrm{d}t$$

Our function is in terms of ωt and $f(\omega t) = 0.2$ between 0 to π. This time we are integrating with respect to ωt and *not* just *t*. Therefore we have

$$\begin{aligned}
B_k &= \frac{2}{\pi}\int_0^{\pi}\left[f(\omega t)\sin(k\omega t)\right]\mathrm{d}(\omega t) \\
&= \frac{2}{\pi}\int_0^{\pi}\left[0.2\sin(k\omega t)\right]\mathrm{d}(\omega t) && \left[\text{Substituting } f(\omega t) = 0.2\right] \\
&= \frac{0.4}{\pi}\left[-\frac{\cos(k\omega t)}{k}\right]_{\omega t=0}^{\omega t=\pi} && \left[\text{Integrating by } \int \sin(kx)\,\mathrm{d}x = -\frac{\cos(kx)}{k}\right] \\
&= -\frac{0.4}{k\pi}\left[\cos(k\pi) - \underbrace{\cos(0)}_{=1}\right] && \left[\begin{array}{l}\text{Substituting limits for } \omega t \\ \text{and taking out a factor of } -1/k\end{array}\right] \\
B_k &= -\frac{0.4}{k\pi}\left[\cos(k\pi) - 1\right]
\end{aligned}$$

Recall the following trigonometric result for $\cos(k\pi)$ which we stated earlier:

$$\cos(k\pi) = \begin{cases} 1 & \text{if } k = \text{even} \\ -1 & \text{if } k = \text{odd} \end{cases}$$

What is B_k equal to when k is even?

Substituting $\cos(k\pi) = 1$ into the last line of the above derivation gives

$$B_k = -\frac{0.4}{k\pi}\left[\cos(k\pi) - 1\right] = -\frac{0.4}{k\pi}\left[1 - 1\right] = 0$$

Hence there are *no* even sine terms.

What is B_k equal to when k is odd?

Substituting $\cos(k\pi) = -1$ into the last line of the above derivation gives

$$\begin{aligned}
B_k &= -\frac{0.4}{k\pi}\left[\cos(k\pi) - 1\right] \\
&= -\frac{0.4}{k\pi}\left[-1 - 1\right] = -\frac{0.4}{k\pi}\left[-2\right] = \frac{0.8}{k\pi}
\end{aligned}$$

The first few odd values of the sine coefficients are given by substituting *odd* values of *k* into the above result, $B_k = \frac{0.8}{k\pi}$:

$$B_1 = \frac{0.8}{\pi},\ B_3 = \frac{0.8}{3\pi},\ B_5 = \frac{0.8}{5\pi},\ B_7 = \frac{0.8}{7\pi},\ \cdots$$

How do we find the Fourier series of $f(\omega t)$?

By substituting $A_0 = 0,\ A_k = 0$ and the above B_k values into the generic Fourier series:

(17.2) $$f(t) = A_0 + A_1\cos(t) + A_2\cos(2t) + \ldots + B_1\sin(t) + B_2\sin(2t) + \ldots$$

Remember our function is in terms of ωt:

$$f(\omega t) = \underbrace{0}_{\text{Constant term}} + \underbrace{0}_{\text{Cosine terms}} + \underbrace{\frac{0.8}{\pi}\sin(\omega t) + 0 + \frac{0.8}{3\pi}\sin(3\omega t) + 0 + \frac{0.8}{5\pi}\sin(5\omega t) + 0 + \cdots}_{\text{Sine terms}}$$

$$= \frac{0.8}{\pi}\sin(\omega t) + \frac{0.8}{3\pi}\sin(3\omega t) + \frac{0.8}{5\pi}\sin(5\omega t) + \cdots$$

$$= \frac{0.8}{\pi}\left[\sin(\omega t) + \frac{\sin(3\omega t)}{3} + \frac{\sin(5\omega t)}{5} + \cdots\right] \qquad \left[\begin{array}{l}\text{Taking out the}\\ \text{common factor } 0.8/\pi\end{array}\right]$$

This is the Fourier series of the rectangular waveform shown in Fig. 31.

Note that the Fourier series of $f(\omega t)$ only contains odd sine coefficients.

If we substitute $\omega t = \frac{\pi}{4}$ into this Fourier series then we obtain a beautiful series result.

If $\omega t = \frac{\pi}{4}$ then by the graph of Fig. 31 we have $f\left(\frac{\pi}{4}\right) = 0.2$, and so we have

$$0.2 = \frac{0.8}{\pi}\left[\sin\left(\frac{\pi}{4}\right) + \frac{\sin(3\pi/4)}{3} + \frac{\sin(5\pi/4)}{5} + \frac{\sin(7\pi/4)}{7} + \ldots\right]$$

$$\frac{0.2\pi}{0.8} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(\frac{1}{3}\right) - \frac{1}{\sqrt{2}}\left(\frac{1}{5}\right) - \frac{1}{\sqrt{2}}\left(\frac{1}{7}\right) + \cdots \qquad \left[\begin{array}{l}\text{Evaluating the sine}\\ \text{of mutliples of } \pi/4\end{array}\right]$$

$$\frac{\pi}{4} = \frac{1}{\sqrt{2}}\left(1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \cdots\right) \qquad [\text{Factorizing}]$$

From this we can write $\pi/4$ as the infinite series:

$$\frac{\pi}{4} = \frac{1}{\sqrt{2}}\left(1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

You will find the following properties of odd and even functions helpful:

$$(\text{odd function}) \times (\text{even function}) = \text{odd function}$$
$$(\text{even function}) \times (\text{even function}) = \text{even function}$$
$$(\text{odd function}) \times (\text{odd function}) = \text{even function}$$

SUMMARY

If $f(t)$ is an odd function then the Fourier series will have *no* cosine terms and the constant term is also zero. Additionally the sine coefficients are given by

(17.13) $$B_k = \frac{2}{\pi}\int_0^{\pi} f(t)\sin(kt)\,\mathrm{d}t$$

If $f(t)$ is an even function then the Fourier series will have *no* sine terms. Additionally the constant and cosine coefficients are given by

(17.14) $$A_0 = \frac{1}{\pi}\int_0^{\pi} f(t)\,\mathrm{d}t$$ [Constant term]

(17.15) $$A_k = \frac{2}{\pi}\int_0^{\pi} f(t)\cos(kt)\,\mathrm{d}t$$ [Cosine coefficients]