SECTION F **Half Range Fourier Series**

By the end of this section you will be able to

- find a Fourier series expansion of a function defined over a finite interval

F1 **Extending Definition**

In this section we examine functions that are defined for a finite interval. For example the following function is only defined in the interval $0 < t < \pi$:

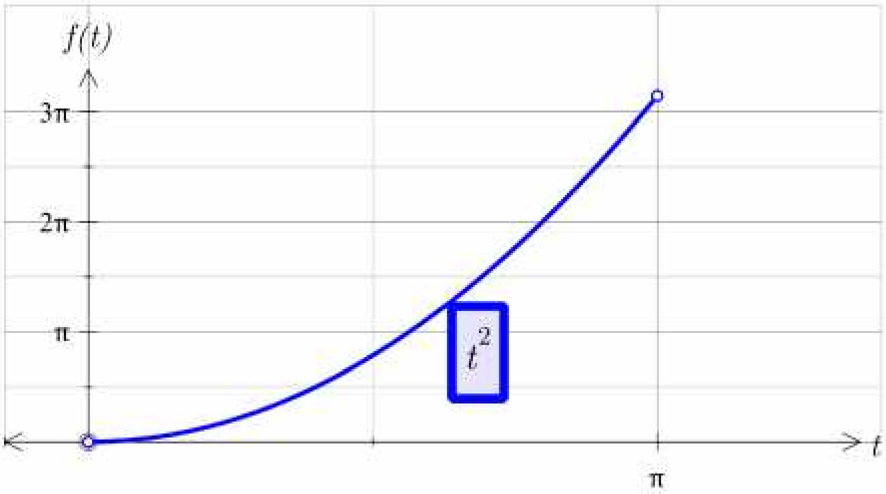


Figure 44

Clearly this function *cannot* be periodic as periodic functions have *infinite* interval.

So naturally we ask, is it possible to find Fourier series of such functions?

Yes but we need to convert the function into a periodic function. This means that we can find Fourier series of non-periodic functions such as e^t provided we consider a finite interval.

We extend the definition so that the function is periodic because the Fourier series only applies to periodic functions.

The most common extension is the **half range Fourier series**.

Generally the function is defined in the interval 0 to L which is half the interval of $-L$ to L, hence the term half-range. For example consider the function

$$f\left(t\right) = 1 \qquad 0 < t < \pi$$

where the interval, $0 < t < \pi$, is half the period of 2π. Hence we need to define $f\left(t\right)$ for the interval $-\pi < t < 0$. We can extend the definition of $f\left(t\right)$ so that the Fourier series of $f\left(t\right)$ ONLY has certain terms as the next example demonstrates.

Example 13

Let

$$f(t) = 1 \qquad \text{when } \; 0 < t < \pi$$

Extend the definition of the function $f(t)$ so that it has a period 2π and the resulting Fourier series has

(i) ONLY sine terms

(ii) ONLY cosine terms

(iii) BOTH sine and cosine terms

Solution

We first sketch $f(t)$:

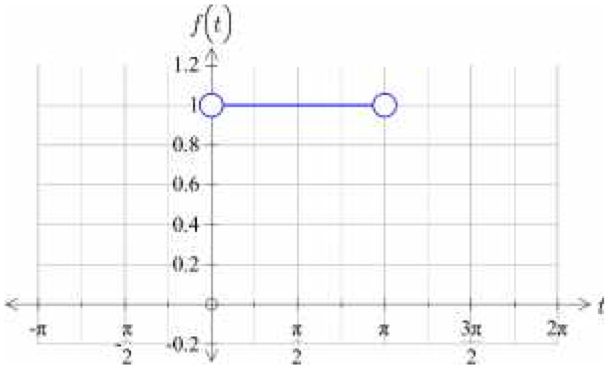


Figure 45

(i) The function has ONLY sine terms if the function is ODD, that is it is symmetrical about the origin:

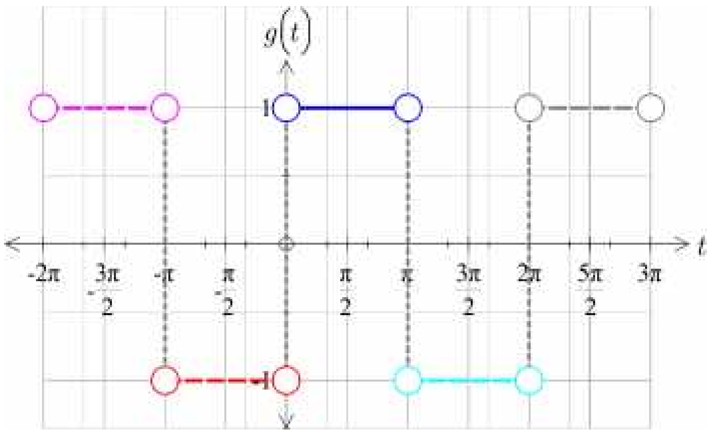


Figure 46

We can write this function with period 2π as

$$g(t) = \begin{cases} 1 & 0 < t < \pi \\ -1 & -\pi < t < 0 \end{cases}$$

(ii) The Fourier series has ONLY cosine terms if the given function is EVEN, that is it is symmetrical about the vertical axis.

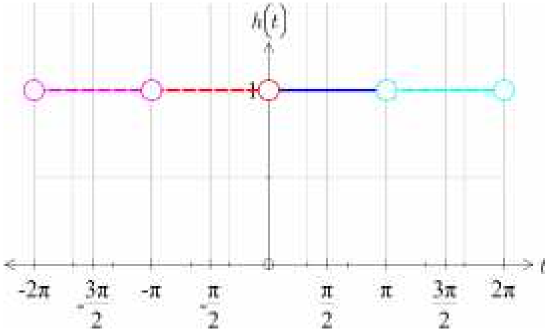


Figure 47

Clearly the function $h(t)$ has period 2π and is given by

$$h(t)=\begin{cases}1 & 0<t<\pi\\ 1 & -\pi<t<0\end{cases}$$

(iii) The Fourier series contains both, sine and cosine, terms if the given function is neither ODD nor EVEN. It is **not** symmetrical about the vertical axis or the origin. We construct the following waveform:

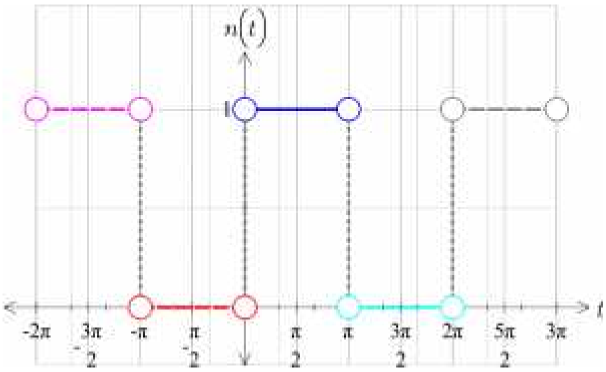


Figure 48

We write this with period 2π as

$$n(t)=\begin{cases}1 & 0<t<\pi\\ 0 & -\pi<t<0\end{cases}$$

The zero in the second line of $g(t)$ could be any value apart from -1 and 1 because these are odd and even functions respectively as you can see from parts (a) and (b).

The three periodic functions, $g(t)$, $h(t)$ and $n(t)$, of this example will have different Fourier series (see below) but *all* three Fourier series will represent the given function

$$f(t)=1 \qquad \text{when} \quad 0<t<\pi$$

We have found the Fourier series of these functions in previous sections or Exercise 17f in the case of $h(t)$ and they are:

$$g(t)=\frac{4}{\pi}\left[\sin(t)+\frac{\sin(3t)}{3}+\frac{\sin(5t)}{5}+\ldots\right] \qquad \text{[See Example 4 on section 17B]}$$

$$h(t)=1 \quad \text{[You are asked to find this in Exercise 17f.]}$$

$$n(t)=\frac{1}{2}+\frac{2}{\pi}\left[\sin(t)+\frac{\sin(3t)}{3}+\frac{\sin(5t)}{5}+\ldots\right]$$ [This is question 1 of Exercise 17b.]

Note the different Fourier series but they should approximate to $f(t)=1$ between 0 and π.

Below is the Graphical representation of the Fourier series of $g(t)$, $h(t)$ and $n(t)$ in Maple:

> $g := t \to \frac{4}{\pi}\left(\sin(t)+\frac{\sin(3\,t)}{3}+\frac{\sin(5\,t)}{5}+\frac{\sin(7\,t)}{7}\right)$

$$t \to \frac{4\left(\sin(t)+\frac{1}{3}\sin(3\,t)+\frac{1}{5}\sin(5\,t)+\frac{1}{7}\sin(7\,t)\right)}{\pi}$$

> $h := t \to 1$

$$t \to 1$$

> $n := t \to \frac{1}{2}+\frac{2}{\pi}\left(\sin(t)+\frac{\sin(3\,t)}{3}+\frac{\sin(5\,t)}{5}\right)$

$$t \to \frac{1}{2}+\frac{2\left(\sin(t)+\frac{1}{3}\sin(3\,t)+\frac{1}{5}\sin(5\,t)\right)}{\pi}$$

> $plot(\{g, h, n\}, -2\,\pi..2\,\pi)$

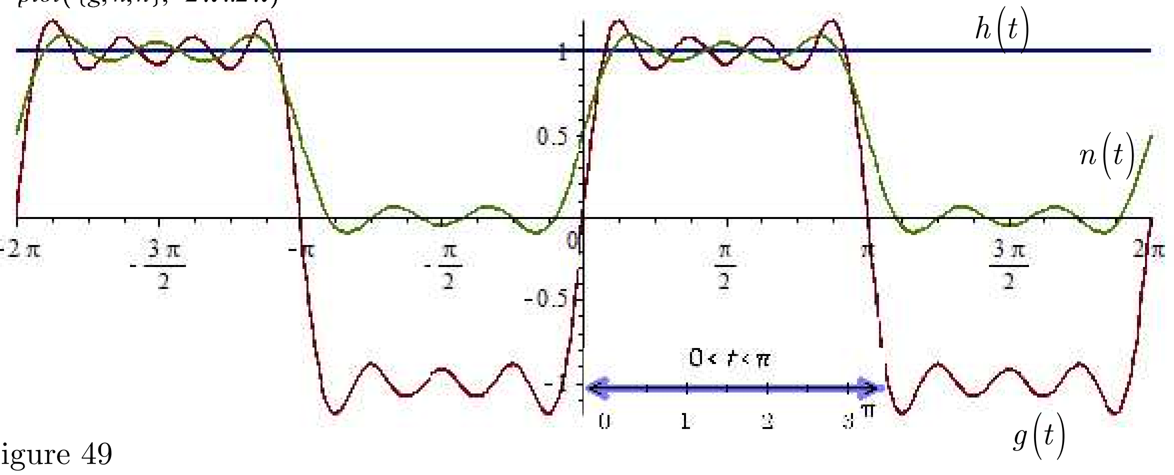


Figure 49

Note that the graph approximates $f(t)$ between 0 and π.

Consider another example:

In the exercises 17f you are asked to look at the extension of

$$f(t)=t \qquad \text{if} \quad 0 \le t < \pi$$

So that it has only

(a) sine terms (b) cosine terms (c) both sine and cosine terms

We obtain the following Fourier series

(a) $f(t)=2\left[\sin(t)-\frac{\sin(2t)}{2}+\frac{\sin(3t)}{3}-\frac{\sin(4t)}{4}+\frac{\sin(5t)}{5}-\cdots\right]$

(b) $f(t) = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos(t) + \frac{\cos(3t)}{3^2} + \frac{\cos(5t)}{5^2} + \frac{\cos(7t)}{7^2} + \cdots\right]$

(c) $f(t) = \frac{\pi}{4} - \frac{2}{\pi}\left[\cos(t) + \frac{\cos(3t)}{3^2} + \frac{\cos(5t)}{5^2} + \cdots\right] + \left[\sin(t) - \frac{\sin(2t)}{2} + \frac{\sin(3t)}{3} - \cdots\right]$

By using Maple we can see the graphical output of all three graphs together as follows:

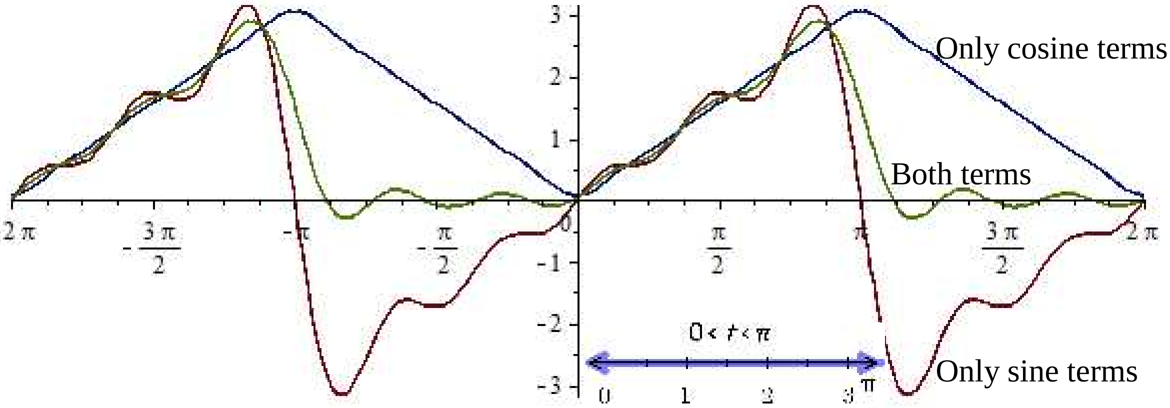


Figure 50

Note the different graphs as they have different Fourier series but between 0 and π they all approximate to $f(t) = t$.

Example 14

Obtain the half-range Fourier sine series of the function:

$$f(t) = 2t \qquad 0 \le t < \pi$$

Solution

For ONLY sine terms we have to make $f(t)$ into an ODD function. Extending $f(t)$ so that $f(t)$ is odd we have

$$g(t) = 2t \qquad -\pi < t < \pi$$

because $2t$ is an odd function.

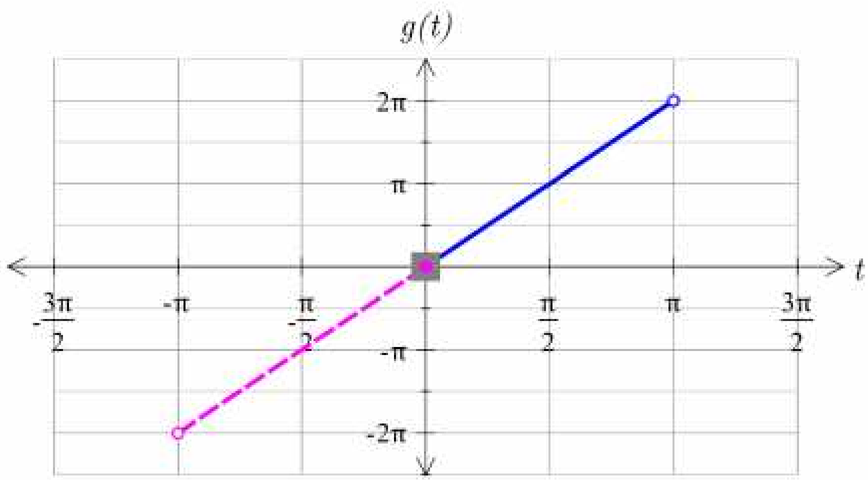


Figure 51

How do we find the Fourier series?

The Fourier series only has sine terms and the coefficients are given by

$$
\begin{aligned}
B_k &= \frac{2}{\pi}\int_0^{\pi}\left[f(t)\sin(kt)\right]\,\mathrm{d}t \\
&= \frac{2}{\pi}\int_0^{\pi}\left[2t\sin(kt)\right]\,\mathrm{d}t \qquad \left[\text{Substituting } f(t)=2t\right] \\
&= \frac{4}{\pi}\int_0^{\pi}\left[t\sin(kt)\right]\,\mathrm{d}t \qquad (*)
\end{aligned}
$$

How do we integrate $\int_0^{\pi}\left[t\sin(kt)\right]\,\mathrm{d}t$ *?*

Use integration by parts with:

$$u = t \qquad v' = \sin(kt)$$

$$u' = 1 \quad [\text{Differentiating}] \qquad v = \int \sin(kt)\,\mathrm{d}t = -\frac{\cos(kt)}{k}$$

Substituting this into the integration by parts formula gives:

$$
\begin{aligned}
\int_0^{\pi}\left[t\sin(kt)\right]\,\mathrm{d}t &= uv - \int u'v\,\mathrm{d}t \\
&= -\left[\frac{t\cos(kt)}{k}\right]_0^{\pi} - \int_0^{\pi}(1)\left(-\frac{\cos(kt)}{k}\right)\mathrm{d}t \\
&= -\left[\frac{\pi\cos(k\pi)}{k} - 0\right] + \left[\frac{\sin(kt)}{k^2}\right]_0^{\pi} \\
&= -\frac{\pi\cos(k\pi)}{k} + \underbrace{\left[\frac{\sin(k\pi)-\sin(0)}{k^2}\right]}_{=0} \qquad \left[\begin{array}{l}\text{Because} \\ \sin(k\pi)=\sin(0)=0\end{array}\right]
\end{aligned}
$$

$$\int_0^{\pi}\left[t\sin(kt)\right]\,\mathrm{d}t = -\frac{\pi\cos(k\pi)}{k}$$

Substituting this $\int_0^{\pi}\left[t\sin(kt)\right]\,\mathrm{d}t = -\frac{\pi\cos(k\pi)}{k}$ into (*) gives

$$B_k = \frac{4}{\cancel{\pi}}\left[-\frac{\cancel{\pi}\cos(k\pi)}{k}\right] = -\frac{4}{k}\cos(k\pi)$$

If k is odd then $\cos(k\pi) = -1$ and we have

$$B_k = -\frac{4}{k}(-1) = \frac{4}{k} \quad (\text{odd } k)$$

If k is even then $\cos(k\pi) = 1$ and we have

$$B_k = -\frac{4}{k}(1) = -\frac{4}{k} \quad (\text{even } k)$$

Hence the Fourier series is given by

$$g(t) = \underbrace{0}_{\text{constant}} + \underbrace{0}_{\text{No cosine terms}} + 4\sin(t) - \frac{4\sin(2t)}{2} + \frac{4\sin(3t)}{3} - \frac{4\sin(4t)}{4} + \dots$$

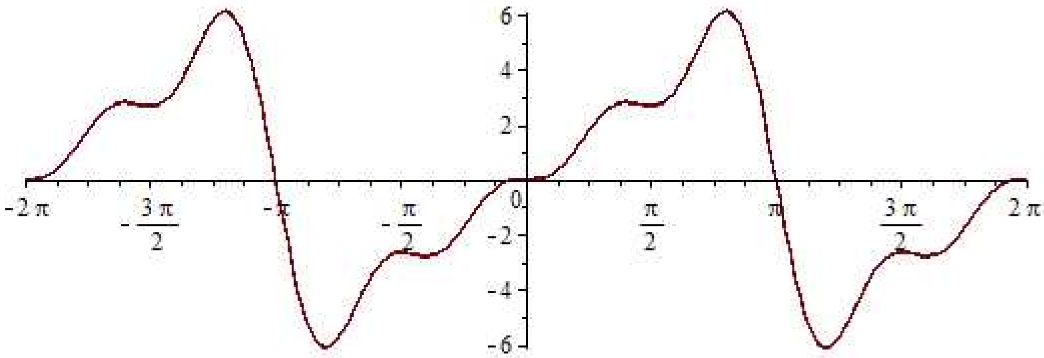
$$= 4\left[\sin(t) - \frac{\sin(2t)}{2} + \frac{\sin(3t)}{3} - \frac{\sin(4t)}{4} + \dots\right] \qquad \left[\text{Taking out } 4\right]$$

We can see the graphical representation of this Fourier series by using the computer algebra system Maple:

$> f := t \to 4\left(\sin(t) - \frac{\sin(2t)}{2} + \frac{\sin(3t)}{3} - \frac{\sin(4t)}{4}\right)$

$$t \to 4\sin(t) - 2\sin(2t) + \frac{4}{3}\sin(3t) - \sin(4t)$$

$> plot(f, -2\pi..2\pi)$



The first 4 non-zero terms of the Fourier series of the given waveform $f(t)$.

Figure 52

You are asked to find the cosine expansion of this sawtooth function in Exercise 17f.

Why extend the given function of Example 14; $f(t) = 2t,\ 0 \le t < \pi$ *?*

As you can see from Fig. 52 that the above extension of $f(t)$ gives a sawtooth waveform which is normally used in television for tracing the pixels on the screen. Between 0 and π it is just like the function $y = 2x$.

SUMMARY

Half range series is generally defined in the interval $0 < t < L$ and we can extend this definition by specifying whether we want a sine or cosine series or neither.