SECTION B **Fourier Series**

By the end of this section you will be able to

- determine the Fourier coefficients
- obtain the Fourier series of a given periodic function

B1 **Fourier Coefficients**

In the last section we stated that a function f under certain (Dirichlet) conditions with period 2π can be expressed as

(17.2) $$f(t) = A_0 + A_1\cos(t) + A_2\cos(2t) + \cdots + B_1\sin(t) + B_2\sin(2t) + \cdots$$

where $A_0,\ A_1,\ A_2, \ldots, B_1,\ B_2,\ B_3, \ldots$ are real coefficients.

(17.2) is called a **trigonometric** series and $A_k,\ B_k$ are the kth coefficients of the series where k is a positive whole number.

Most students find understanding mathematical notation a difficult concept. Remember mathematics is like learning a new language and it is important that you understand notation. *What does (17.2) mean?*

It means that periodic functions $f(t)$ of period 2π can be written as an infinite sum of sines and cosines. For example if we are asked to express the function $f(t)$ with period 2π given by

$$f(t) = \begin{cases} 0 & -\pi < t < 0 \\ 3t & 0 < t < \pi \end{cases}$$

as a sum of sines and cosines we get

(*) $$f(t) = \frac{3\pi}{4} - \frac{6}{\pi}\cos(t) - \frac{2}{9\pi}\cos(3t) - \frac{6}{25\pi}\cos(5t) - \ldots + 3\sin(t) - \frac{3}{2}\sin(2t) + \sin(3t) - \ldots$$

(You're asked to show this in Exercises 17c).

By comparing formula (17.2) and (*) what is the value of A_0 *?*

Clearly A_0 is the term *without* cosine and sine attached to it, therefore $A_0 = \frac{3\pi}{4}$.

What are the values of coefficients $A_1,\ A_2,\ A_3,\ A_4$ *and* A_5 *?*

$A_1 = -\dfrac{6}{\pi}$ because this is the number attached to $\cos(t)$. $A_2 = 0$ because there is *no* $\cos(2t)$ term in (*). Similarly $A_3 = -\dfrac{2}{9\pi}$, $A_4 = 0$ [*No* $\cos(4t)$ term in (*)] and $A_5 = -\dfrac{6}{25\pi}$.

What are B_1, B_2 and B_3 equal to?

The B coefficients correspond to the sine terms; that is B_1 represents the coefficient of $\sin(t)$, B_2 represents the coefficient of $\sin(2t)$ and B_3 represents the coefficient of $\sin(3t)$. Hence

$$B_1 = 3, \ B_2 = -\frac{3}{2} \ \text{ and } \ B_3 = 1$$

In this chapter our aim is to find the values of these real coefficients:

$$A_0, \ A_1, \ A_2, \dots, \ B_1, \ B_2, \dots$$

given a periodic function $f(t)$. If $f(t)$ has period 2π then these coefficients are given by the following formulae:

(17.3) $$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t$$ [Average value of $f(t)$]

(17.4) $$A_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,\mathrm{d}t$$ [2 × the average value of $f(t)\cos(kt)$]

(17.5) $$B_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,\mathrm{d}t$$ [2 × the average value of $f(t)\sin(kt)$]

The derivations of these formulae are given as questions in Exercise 17(b).

A_0, A_k and B_k are called the **Fourier coefficients**. With A_0, A_k and B_k replaced in (17.2) by the above then (17.2) is called a **Fourier series** of $f(t)$.

It is useful to remember what these coefficients represent:

A_0 is the average value of $f(t)$ over one complete period.

A_k is 2 × [the average value of $f(t)\cos(kt)$ over one complete period.]

B_k is 2 × [the average value of $f(t)\sin(kt)$ over one complete period.]

You may be asking yourself why there is an A_0 term but no B_0 term? Well, look at (17.5) given above, when $k = 0$ the right hand side is 0 since $\sin(0) = 0$, this means that B_0 will always be zero and so never needs to be calculated!

B2 **Dirichlet Conditions**

We hinted at the start that we can only express a periodic function $f(t)$ as a Fourier series if it meets certain criteria. These criteria are given formally as follows.

Let $f(t)$ be a periodic function then $f(t)$ can be expressed as a Fourier series provided that

(1) there are only a finite number of discontinuities over a complete period

(2) $\int |f(t)| \, \mathrm{d}t$ is finite over a complete period

(3) $f(t)$ has a finite average value

Conditions (1), (2) and (3) are called the **Dirichlet** conditions and many engineering functions satisfy these. We can use them as a quick check before embarking on trying to find the Fourier series of a function. If any of the above are not satisfied by the function then we *cannot* find the Fourier series of the function.

B3 **Fourier Series**

Finding the Fourier series of a periodic function is a long and challenging process. It involves knowing and applying trigonometric results, integration and algebra. It is very easy to make a slip in the calculation of establishing a Fourier series. You need to be very careful with the signs because this is where most students slip up. Also you need to know the integration of $\sin(kt)$ and $\cos(kt)$ where k is a whole number.

Some trigonometric results need to be second nature to you such as:

(†) $\sin(-x) = -\sin(x)$

(††) $\cos(-x) = \cos(x)$

(**) $\sin(k\pi) = 0$ where k is a whole number

This last result means $\sin(\text{an integer multiple of } \pi) = 0$ which we can see from:

$\sin(k\pi) = 0$

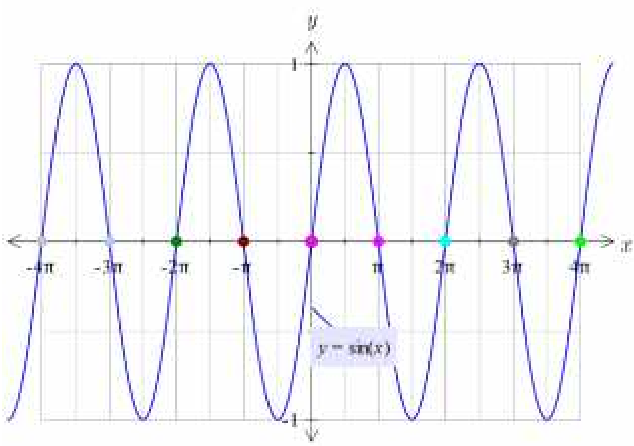


Figure 10(a)

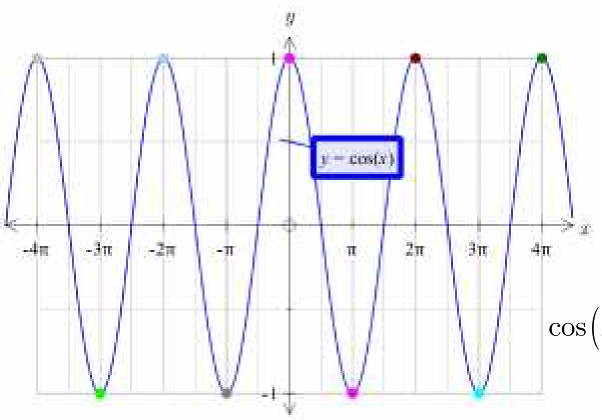
We also have

(***) $\cos(k\pi) = \begin{cases} 1 & \text{if } k \text{ is even} \\ -1 & \text{if } k \text{ is odd} \end{cases}$

This result means that

$$\cos\left(\text{an even multiple of } \pi\right) = 1$$
$$\cos\left(\text{an odd multiple of } \pi\right) = -1$$

We can visualize these from the following graph:



$$\cos\left(k\pi\right) = \begin{cases} 1 & \text{if } k \text{ is even} \\ -1 & \text{if } k \text{ is odd} \end{cases}$$

Figure 10(b)

We will apply these results in Example 4.

Example 4

(i) Sketch the following square waveform:

$$f\left(t\right) = \begin{cases} 1 & 0 < t < \pi \\ -1 & \pi < t < 2\pi \end{cases}$$

with period 2π.

(ii) Determine the Fourier series for $f\left(t\right)$.

Solution

(i) *What does the graph of $f\left(t\right)$ look like?*

Decoding the function notation gives us that the function $f\left(t\right) = 1$ for t between 0 and π and $f\left(t\right) = -1$ for t between π and 2π.

What does a function with period 2π mean?

This means that the function repeats itself every 2π interval. Hence combining both answers gives us the following sketch of the function $f\left(t\right)$.

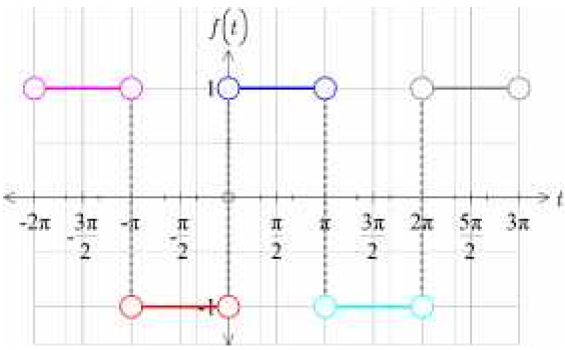


Figure 11

(ii) *What are we trying to find?*

The values of the coefficients A_0, A_k and B_k given in (17.2). *How do we find A_0 ?*

By using the above formula:

(17.3) $$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t$$

By examining the graph of Fig. 11 we can split $f(t)$ into $f(t) = 1$ for t between 0 and π and $f(t) = -1$ for t between $-\pi$ and 0. We have

$$\begin{aligned}
A_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t \\
&= \frac{1}{2\pi}\left\{\int_0^{\pi}(1)\,\mathrm{d}t + \int_{-\pi}^{0}(-1)\,\mathrm{d}t\right\} && \left[\text{Splitting } f(t)\right] \\
&= \frac{1}{2\pi}\left\{\left[t\right]_0^{\pi} - \left[t\right]_{-\pi}^{0}\right\} && \left[\text{Integrating}\right] \\
&= \frac{1}{2\pi}\left\{\left[\pi - 0\right] - \left[0 - (-\pi)\right]\right\} && \left[\text{Substituting for } t\right] \\
&= \frac{1}{2\pi}\underbrace{\left\{\pi - \pi\right\}}_{=0} && \left[\text{Simplifying}\right] \\
A_0 &= 0
\end{aligned}$$

From Fig. 11, notice that this $A_0 = 0$ is the average value of the function $f(t)$ over a complete period.

How do we find the value of A_k ?

Similar process to finding A_0 but this time we use

(17.4) $$A_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,\mathrm{d}t$$

Additionally it is longer and there is more integration involved because of $\cos(kt)$.

Again we split $f(t)$ into 1 and -1.

$$\begin{aligned}
A_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,\mathrm{d}t \\
&= \frac{1}{\pi}\left\{\int_0^{\pi}(1)\cos(kt)\,\mathrm{d}t + \int_{-\pi}^{0}(-1)\cos(kt)\,\mathrm{d}t\right\} && \left[\text{Replacing } f(t)\right] \\
&= \frac{1}{\pi}\left\{\int_0^{\pi}\cos(kt)\,\mathrm{d}t - \int_{-\pi}^{0}\cos(kt)\,\mathrm{d}t\right\} \\
&= \frac{1}{\pi}\left\{\left[\frac{\sin(kt)}{k}\right]_0^{\pi} - \left[\frac{\sin(kt)}{k}\right]_{-\pi}^{0}\right\} && \left[\text{Integrating by } \int\cos(kt)\,\mathrm{d}t = \frac{\sin(kt)}{k}\right] \\
&= \frac{1}{k\pi}\left\{\left[\sin(kt)\right]_0^{\pi} - \left[\sin(kt)\right]_{-\pi}^{0}\right\} && \left[\text{Taking out a factor of } \frac{1}{k}\right] \\
&= \frac{1}{k\pi}\left\{\left[\underbrace{\sin(k\pi)}_{=0 \text{ by } (**)} - \sin(0)\right] - \left[\sin(0) - \underbrace{\sin(k(-\pi))}_{=0 \text{ by } (**)}\right]\right\} && \left[\begin{array}{l}\text{Substituting the limits} \\ \text{and } (**)\ \sin(k\pi) = 0\end{array}\right] \\
&= \frac{1}{k\pi}\left\{\left[0 - 0\right] - \left[0 - 0\right]\right\} = 0
\end{aligned}$$

What does $A_k = 0$ *represent?*

It is twice the average value of the function $f(t)\cos(kt)$. We can visualize this for $k = 1$, that is $A_1 = 0$ because the graph of this is given by:

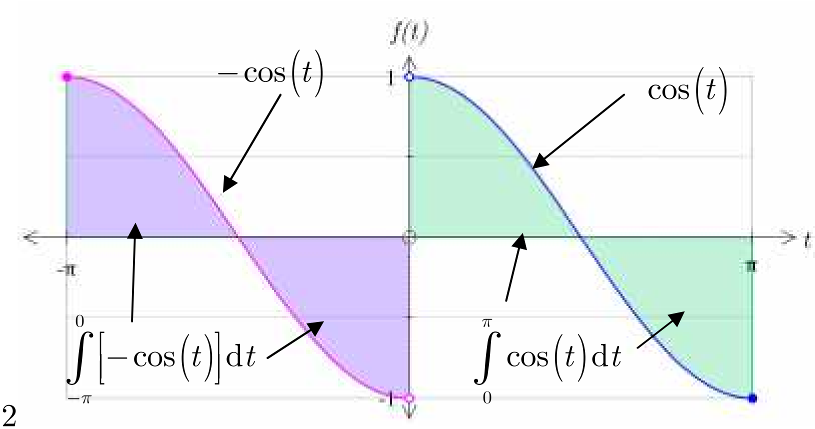


Figure 12

The average value of this function $f(t)\cos(t)$ in Fig. 12 is 0 so $A_1 = 2 \times 0 = 0$.

Similarly $A_2 = A_3 = \cdots = A_k = 0$.

Don't be surprised by the answers of zeros for A_k and A_0. There can be quite a few zeros in the Fourier series. *How do we find* B_k *?*

Similarly by applying the formula

$$B_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,\mathrm{d}t \qquad (17.5)$$

Splitting $f(t)$ into 1 and -1 gives:

$$B_k = \frac{1}{\pi}\left\{\int_0^{\pi}(1)\sin(kt)\,\mathrm{d}t + \int_{-\pi}^{0}(-1)\sin(kt)\,\mathrm{d}t\right\} \qquad \left[\text{Replacing } f(t)\right]$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi}\sin(kt)\,\mathrm{d}t - \int_{-\pi}^{0}\sin(kt)\,\mathrm{d}t\right\}$$

$$= \frac{1}{\pi}\left\{\left[-\frac{\cos(kt)}{k}\right]_0^{\pi} - \left[-\frac{\cos(kt)}{k}\right]_{-\pi}^{0}\right\} \qquad \left[\text{Integrating by } \int \sin(kt)\,\mathrm{d}t = -\frac{\cos(kt)}{k}\right]$$

$$= -\frac{1}{k\pi}\left\{\left[\cos(kt)\right]_0^{\pi} - \left[\cos(kt)\right]_{-\pi}^{0}\right\} \qquad \left[\text{Taking out a factor of } -\frac{1}{k}\right]$$

$$= -\frac{1}{k\pi}\left\{\left[\cos(k\pi) - \cos(0)\right] - \left[\cos(0) - \cos(k(-\pi))\right]\right\} \qquad \left[\begin{array}{l}\text{Substituting} \\ \text{the limits}\end{array}\right]$$

$$= -\frac{1}{k\pi}\left\{\cos(k\pi) - 1 - 1 + \cos(-k\pi)\right\} \qquad \left[\text{Remember } \cos(0) = 1\right]$$

$$= -\frac{1}{k\pi}\left\{\cos(k\pi) - 2 + \underbrace{\cos(k\pi)}_{\text{By } (\dagger\dagger)}\right\} \qquad \left[(\dagger\dagger) \quad \cos(-x) = \cos(x)\right]$$

$$B_k = -\frac{1}{k\pi}\left\{2\cos(k\pi) - 2\right\} \qquad \left[\text{Collecting } \cos(k\pi) \text{ terms}\right]$$

If k is even then $\cos(k\pi) \underbrace{=}_{\text{By } (***)} 1$, substituting this into the last line above gives:

$$B_k = -\frac{1}{k\pi}\left\{2\left(1\right)-2\right\} = 0$$

Another zero. If k is odd then $\cos\left(k\pi\right) \underbrace{=}_{\text{By (***)}} -1$, again substituting this into the last line in the above derivation gives

$$B_k = -\frac{1}{k\pi}\left\{2\left(-1\right)-2\right\} = -\frac{1}{k\pi}\left\{-4\right\} = \frac{4}{k\pi}$$

Putting all these coefficients together we have

$$A_0 = 0$$
$$A_k = 0$$
$$B_k = \begin{cases} 0 & \text{if } k \text{ is even} \\ 4 / k\pi & \text{if } k \text{ is odd} \end{cases}$$

What do we do next?

Substitute these into

(17.2) $\quad f\left(t\right) = A_0 + A_1\cos\left(t\right) + A_2\cos\left(2t\right) + \cdots + B_1\sin\left(t\right) + B_2\sin\left(2t\right) + \cdots$

The *only* non-zero values are the odd sine terms:

$$\begin{aligned} f\left(t\right) &= \underbrace{0}_{\text{Constant term}} + \underbrace{0+0+\ldots+0}_{\text{No cosine terms}} + \frac{4}{\pi}\sin\left(t\right) + 0 + \frac{4}{3\pi}\sin\left(3t\right) + 0 + \frac{4}{5\pi}\sin\left(5t\right) + \ldots \\ &= \frac{4}{\pi}\sin\left(t\right) + \frac{4}{3\pi}\sin\left(3t\right) + \frac{4}{5\pi}\sin\left(5t\right) + \ldots \qquad \left[\text{Simplifying}\right] \\ &= \frac{4}{\pi}\left[\sin\left(t\right) + \frac{\sin\left(3t\right)}{3} + \frac{\sin\left(5t\right)}{5} + \ldots\right] \qquad \left[\begin{array}{l}\text{Taking out a common} \\ \text{factor of } 4/\pi\end{array}\right] \end{aligned}$$

This is the Fourier series for $f\left(t\right)$ which was given by

$$f\left(t\right) = \begin{cases} 1 & 0 < t < \pi \\ -1 & \pi < t < 2\pi \end{cases}$$

This means that we can decompose the square waveform $f\left(t\right)$ as an infinite sum of odd sine terms.

Note that to establish the Fourier series for a simple function such as the one in Example 4 is a long process. It involves a number of different topics such as integration, algebra, trigonometry and sketching graphs . Note how easy it is to make a slip of the signs.

You will make your life much easier in evaluating Fourier series if you notice that terms such as $\int_0^{\pi}\cos\left(kt\right)\mathrm{d}t = \int_{\pi}^{2\pi}\cos\left(kt\right)\mathrm{d}t = \int_{-\pi}^{\pi}\sin\left(kt\right) = 0$. [The average value of trigonometric functions over a complete period is zero.] This reduces the amount of work significantly and can also stop you making basic algebra mistakes.

Why do we write a simple square waveform shown in Fig. 11 as an infinite sum of sines and cosines?

Because it is much easier to calculate how a system such as an amplifier or filter will respond to sine (or cosine) waveforms but difficult to calculate how it will respond to a periodic square wave. We break the difficult periodic function into a series of sine and cosine functions and then calculate the response to this. Eventually we add up all the responses and we get the response to the original function. Often this is the only way to solve a problem of this kind.

Converting square waveforms into a sum of sinusoids (sin and cosine) is useful in seeing how amplifiers and filters behave towards these waveforms.

Also if we consider the square waveform then we *cannot* differentiate it at the jumps. It is easier to think of the square wave as an infinite Fourier series and then ignore some of the terms of the series in order to achieve an approximation.

In general, the Fourier series provides us with a way to perform mathematical analysis on complicated periodic functions because it breaks the function down into a series of numbers and the period.

For the square wave the Fourier series selects the odd integers as you can observe from the previous example:

$$f(t)=\frac{4}{\pi}\left[\sin(t)+\frac{\sin(3t)}{3}+\frac{\sin(5t)}{5}+\ldots\right]\qquad\left[f(t)\ \text{ is a square wave}\right]$$

It's really amazing that a square wave is connected to the reciprocals of the odd integers. Who would have thought that you could express a square waveform as the sin of odd multiples of π?

Another beautiful mathematical result can be obtained from this series by substituting $t=\frac{\pi}{2}$. *What is* $f\left(\frac{\pi}{2}\right)$ *equal to in the given* $f(t)=\begin{cases}1 & 0<t<\pi\\ -1 & \pi<t<2\pi\end{cases}$ *?*

$$f\left(\frac{\pi}{2}\right)=1\qquad\left[\text{Because } 0<\frac{\pi}{2}<\pi\right]$$

Substituting $t=\frac{\pi}{2}$ into the right hand side of $f(t)=\frac{4}{\pi}\left[\sin(t)+\frac{\sin(3t)}{3}+\frac{\sin(5t)}{5}+\ldots\right]$

$$\underbrace{f\left(\frac{\pi}{2}\right)}_{=1}=\frac{4}{\pi}\left[\sin\left(\frac{\pi}{2}\right)+\frac{\sin(3\pi/2)}{3}+\frac{\sin(5\pi/2)}{5}+\frac{\sin(7\pi/2)}{7}+\ldots\right]$$

$$1=\frac{4}{\pi}\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\right]$$

Transposing this gives

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Hence the Fourier series of a square wave gives this pretty result on infinite series.

The Fourier series is also useful in solving differential equations. Consider the following differential equation where the forcing function $F(t)$ is a periodic function:

$$\frac{d^2x}{dt^2} + kx = F(t)$$

This can be a very difficult problem for an arbitrary forcing function $F(t)$, but if we break down $F(t)$ into its Fourier series then we only need to solve the following three simple problems:

$$\frac{d^2x}{dt^2} + kx = A_k \cos(kt)$$

$$\frac{d^2x}{dt^2} + kx = B_k \sin(kt)$$

$$\frac{d^2x}{dt^2} + kx = A_0$$

These three equations are easily solvable.

B4 **Plotting Graphs of Fourier Series**

By using a computer algebra system (MAPLE) we can see that the Fourier series of Example 4 does approximate to the square wave $f(t)$. Below is the MAPLE output for the first 3 non-zero terms of the Fourier series.

> $f := t \to \dfrac{4\left(\sin(t) + \dfrac{\sin(3t)}{3} + \dfrac{\sin(5t)}{5}\right)}{\pi}$

$$f := t \to \frac{4\sin(t) + \frac{4}{3}\sin(3t) + \frac{4}{5}\sin(5t)}{\pi}$$

>$plot(f, -2\pi..2\pi)$

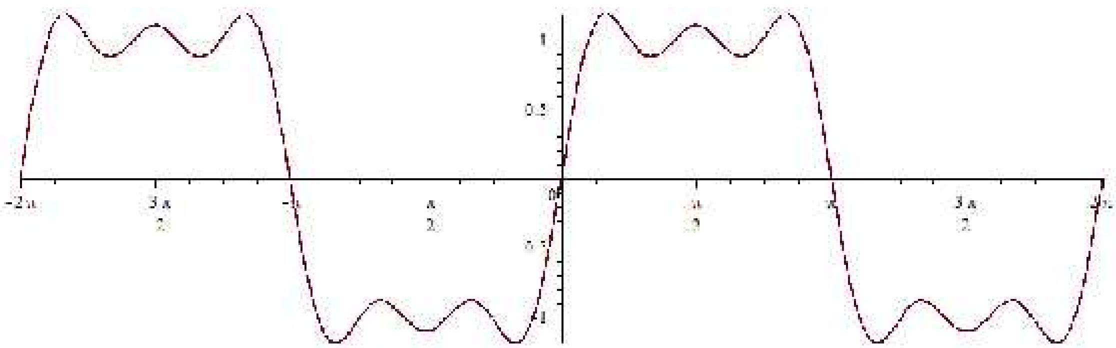


Figure 13(a) The graph of first 3 non-zero terms of the Fourier series of square wave $f(t)$.

To get closer approximation we add more and more terms:

$$> \quad g := t \rightarrow \frac{1}{\pi}\left(4\left(\sin(t) + \frac{\sin(3\,t)}{3} + \frac{\sin(5\,t)}{5} + \frac{\sin(7\,t)}{7} + \frac{\sin(9\,t)}{9} + \frac{\sin(11\,t)}{11}\right)\right)$$

$$g := t \rightarrow \frac{1}{\pi}\left(4\sin(t) + \frac{4}{3}\sin(3\,t) + \frac{4}{5}\sin(5\,t) + \frac{4}{7}\sin(7\,t) + \frac{4}{9}\sin(9\,t) + \frac{4}{11}\sin(11\,t)\right)$$

$> \quad plot(g, -2\,\pi\,..2\,\pi)$

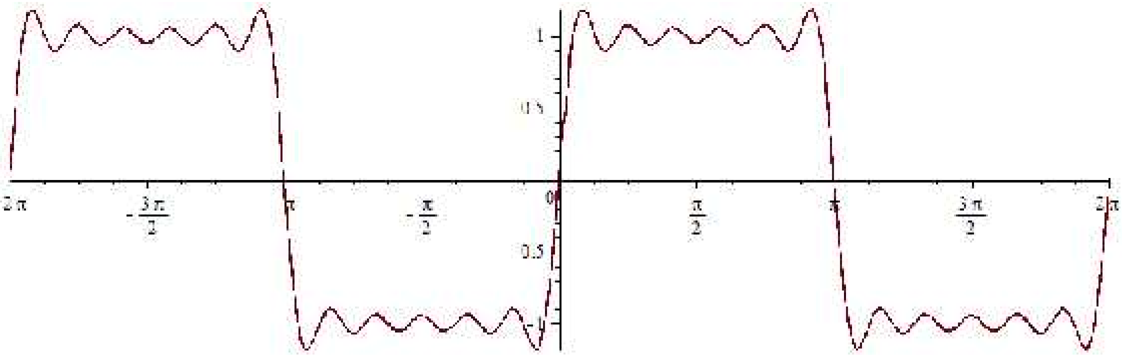


Figure 13(b) The graph of first 6 non-zero terms of the Fourier series of square wave $f(t)$.

To get the square wave

$$f(t) = \begin{cases} 1 & 0 < t < \pi \\ -1 & \pi < t < 2\pi \end{cases}$$

We would have to consider an infinite number of terms. However, what is clear is that adding more terms in the Fourier series results in a more accurate representation of the periodic function $f(t)$.

B5 **Linearity Property of Fourier Series**

Since the Fourier coefficients are defined by integrals and for integrals we have the linearity property:

$$\int \left[cf(t) + dg(t)\right] \mathrm{d}t = c\int f(t)\,\mathrm{d}t + d\int g(t)\,\mathrm{d}t \qquad (c \text{ and } d \text{ are constants})$$

It follows that the same property holds for the Fourier series.

Property 1:

If a periodic function $f(t)$ has a Fourier series given by

$$f(t) = A_0 + A_1\cos(t) + A_2\cos(2t) + \cdots + B_1\sin(t) + B_2\sin(2t) + \cdots$$

Then the Fourier series of $cf(t)$ where c is a constant is

$$cf(t) = cA_0 + cA_1\cos(t) + cA_2\cos(2t) + \cdots + cB_1\sin(t) + cB_2\sin(2t) + \cdots$$

For example given that the Fourier series for

$$f(t)=\begin{cases}1 & 0<t<\pi\\ -1 & \pi<t<2\pi\end{cases}$$

Is $f(t)=\frac{4}{\pi}\left[\sin(t)+\frac{\sin(3t)}{3}+\frac{\sin(5t)}{5}+\ldots\right]$. *What would be the Fourier series for*

$$2f(t)=\begin{cases}2 & 0<t<\pi\\ -2 & \pi<t<2\pi\end{cases}\ ?$$

Well applying the above rule we have

$$2f(t)=2\times\frac{4}{\pi}\left[\sin(t)+\frac{\sin(3t)}{3}+\frac{\sin(5t)}{5}+\ldots\right]=\frac{8}{\pi}\left[\sin(t)+\frac{\sin(3t)}{3}+\frac{\sin(5t)}{5}+\ldots\right]$$

You are asked to show this in Exercise 17b question 2 by using the integration definition. Additionally if the function $g(t)$ of the same period 2π has the Fourier series

$$g(t)=C_0+C_1\cos(t)+C_2\cos(2t)+\cdots+D_1\sin(t)+D_2\sin(2t)+\cdots$$

Then the Fourier series of $cf(t)+dg(t)$ is given by

$$\begin{aligned}cf(t)+dg(t)&=\left[cA_0+cA_1\cos(t)+cA_2\cos(2t)+\cdots+cB_1\sin(t)+cB_2\sin(2t)+\cdots\right]\\&\quad+\left[dC_0+dC_1\cos(t)+dC_2\cos(2t)+\cdots+dD_1\sin(t)+dD_2\sin(2t)+\cdots\right]\\&=\left(cA_0+dC_0\right)+\left(cA_1+dC_1\right)\cos(t)+\left(cA_2+dC_2\right)\cos(2t)+\cdots\\&\quad+\left(cB_1+dD_1\right)\sin(t)+\left(cB_2+dD_2\right)\sin(2t)+\cdots\end{aligned}$$

Property 2:

If $f(t)=A_0+A_1\cos(t)+A_2\cos(2t)+\cdots+B_1\sin(t)+B_2\sin(2t)+\cdots$ and

$$g(t)=C_0+C_1\cos(t)+C_2\cos(2t)+\cdots+D_1\sin(t)+D_2\sin(2t)+\cdots$$

They both have the same period 2π. Then for constants c and d we have

$$\begin{aligned}cf(t)+dg(t)&=\left(cA_0+dC_0\right)+\left(cA_1+dC_1\right)\cos(t)+\left(cA_2+dC_2\right)\cos(2t)+\cdots\\&\quad+\left(cB_1+dD_1\right)\sin(t)+\left(cB_2+dD_2\right)\sin(2t)+\cdots\end{aligned}$$

For example let

$$g(t)=\begin{cases}1 & -\pi<t<0\\ 2 & 0<t<\pi\end{cases}$$

The Fourier series of this is given by

$$\frac{3}{2}+\frac{2}{\pi}\left[\sin(t)+\frac{\sin(3t)}{3}+\frac{\sin(5t)}{5}+\cdots\right]$$

(This is derived in question 3 of Exercise 17b.)

What would be the Fourier series of $2f(t)+3g(t)$ *where* $f(t)$ *is the square wave given in the previous example?*

By the using the second linearity property we have

$$2f(t)+3g(t)=\frac{8}{\pi}\left[\sin(t)+\frac{\sin(3t)}{3}+\frac{\sin(5t)}{5}+\ldots\right]$$

$$+3\left(\frac{3}{2}+\frac{2}{\pi}\left[\sin(t)+\frac{\sin(3t)}{3}+\frac{\sin(5t)}{5}+\cdots\right]\right)$$

$$=3\left(\frac{3}{2}\right)+\left(\frac{8}{\pi}+3\left(\frac{2}{\pi}\right)\right)\sin(t)+\left(\frac{8}{\pi}\left(\frac{1}{3}\right)+3\left(\frac{2}{\pi}\right)\left(\frac{1}{3}\right)\right)\sin(3t)+\left(\frac{8}{\pi}\left(\frac{1}{5}\right)+3\left(\frac{2}{\pi}\right)\left(\frac{1}{5}\right)\right)\sin(5t)+\cdots$$

$$=\frac{9}{2}+\left(\frac{8+6}{\pi}\right)\sin(t)+\frac{1}{3}\left(\frac{8+6}{\pi}\right)\sin(3t)+\frac{1}{5}\left(\frac{8+6}{\pi}\right)\sin(5t)+\cdots$$

$$=\frac{9}{2}+\frac{14}{\pi}\left[\sin(t)+\frac{1}{3}\sin(3t)+\frac{1}{5}\sin(5t)+\cdots\right]$$

These linearity properties are very handy as they reduce the amount of work we have to do to evaluate a Fourier series of complicated functions.

SUMMARY

A period function $f(t)$ can be expressed as

(17.2) $f(t)=A_0+A_1\cos(t)+A_2\cos(2t)+\cdots+B_1\sin(t)+B_2\sin(2t)+\cdots$

To obtain a Fourier series of $f(t)$ with period 2π we replace the coefficients A_0, A_k and B_k by

(17.3) $A_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t$ [Average value of $f(t)$]

(17.4) $A_k=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,\mathrm{d}t$ [2 ×(Average value of $f(t)\cos(kt)$)]

(17.5) $B_k=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,\mathrm{d}t$ [2 ×(Average value of $f(t)\sin(kt)$)]