Section B Properties of Upper and Lower Bounds

By the end of this section you will be able to
- prove properties of the supremum and infimum of a set
- apply properties of supremum and infimum

B1 **Properties of Supremum and Infimum**

In this subsection we prove properties of the supremum and infimum of a set. In general we will prove results for supremum of the set and you are asked to prove the analogous results for the infimum in Exercise 5b.
This is a difficult subsection which requires that you know the definition of the supremum and infimum thoroughly. It might seem like a quantum leap from the material that we have covered so far but proving results always means that you need to understand the work more thoroughly.
Before doing the remainder of this section, learn the definition of the supremum given here again:

Definition (5.2)
Let S be a non-empty subset of real numbers, $\mathbb{R}$, which is bounded above. Then a real number u is the supremum of the set S if and only if
(a) u is an upper bound of the set S
(b) u' is any other upper bound of the set S then $u \le u'$

Remember the supremum is the Least Upper Bound.
For Exercise 5b you will also need to learn the definition of the infimum given in definition (5.3).

Proposition (5.4)
(i) The supremum of a set is unique.
(ii) The infimum of a set is unique.

Proof of (i). We only prove (i). *How?*
We suppose there are two suprema, S and T, and prove they are equal. Here is the proof.
Let A be a non-empty subset of $\mathbb{R}$ which is bounded above. Suppose the real numbers S and T are the suprema of the set A. By the definition of supremum (5.2) (a) both S and T are upper bounds of the set A. By (5.2) (b) we have $S \le T$ and $T \le S$ therefore this gives $S = T$. Hence the supremum of a set is unique.
Proof of (ii). See Exercise 5b.

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Note that we had to use the definition of the supremum, (5.2), to prove the uniqueness of the supremum.
The next proposition and proof is difficult to follow. You have to recall your work from mathematical logic and use the definition of the supremum given in (5.2). Additionally you need to know the definition of upper and lower bounds of a set. It is very easy to get confused on the next proof because of the if and only if statement which means we have to prove the result in both directions, $\Rightarrow$ and $\Leftarrow$.
Also don't let the ε symbol put you off. We use the symbol U to denote the supremum of the non empty set S because it is an Upper bound.
Try following each line carefully.

Proposition (5.5)
Let S be a non-empty subset of real numbers, $\mathbb{R}$. A real number U is the supremum of the set S if and only if it satisfies both the following conditions:
(i) For all $x \in S$ we have $x \le U$
(ii) For every $\varepsilon > 0$ there is a real number y in the set S, $y \in S$, such that

$$U - \varepsilon < y \le U$$

Note. *What does this proposition mean?*
Assume that the real number U is the supremum of the set S then any number less than U is **not** an upper bound of the set S. That is for every positive number ε, however small, the real number $U - \varepsilon$ is **not** an upper bound of the set S. This means that there is an element in the set S, say y, which is greater than $U - \varepsilon$.
Also if U is an upper bound and for every $\varepsilon > 0$ there is a real number $y \in S$ such that $U - \varepsilon < y \le U$ then the real number U is the supremum of the set S.

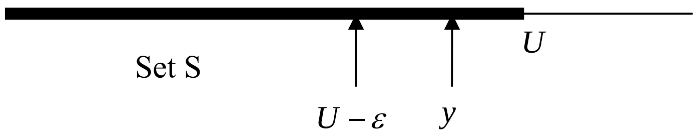

Fig 12
How do we prove this proposition?
Digression: Remember your stuff from chapter 1 on mathematical logic. To prove if and only if statements means that we have to go both ways, for example if *P* and *Q* are statements and we have

P if and only if *Q*

then in symbolic form this is $P \Rightarrow Q$ and $Q \Rightarrow P$.
We first assume *P* and prove *Q* and then we assume *Q* and prove *P*. This is equivalent to $P \Leftrightarrow Q$. If *P* then *Q* and if *Q* then *P*. In this proposition P is the initial statement that the real number U is the supremum of the set S and Q is the given conditions, (i) and (ii).
Proof. We first assume the real number *U* is the supremum of the set S and prove that conditions (i) and (ii) are satisfied. Let U be the supremum, Least Upper Bound, of the set S. By definition of the supremum, (5.2)(a), we have

$$x \le U \qquad \text{For All } x \in S$$

[Because the supremum, U, is an Upper Bound of the set S].
Therefore condition (i) is satisfied. *What else do we need to prove?*
Condition (ii). *How?*
We can try proof by contradiction. Suppose the negation (or not) of condition (ii) is true. *What is the negation of condition (ii)?*
Condition (ii) in symbols says

$$\forall \varepsilon > 0 \quad \exists y \in S \text{ such that } U - \varepsilon < y$$

Negation (or Not) of this is

$$\exists \varepsilon > 0 \quad \forall y \in S \text{ such that } U - \varepsilon \ge y$$

That is suppose there is an $\varepsilon > 0$ for every $y \in S$ such that

$$U - \varepsilon \ge y$$

This means that $U - \varepsilon$ is an upper bound of the set S but $U - \varepsilon < U$ contradicting that U is the supremum of the set *S*. Remember the supremum means it is the Least Upper Bound. This means condition (ii) must be true, so for every $\varepsilon > 0$ there is a real number $y \in S$ such that

$$U-\varepsilon < y \le U$$

We have shown that if U is the supremum of the set S then both the given conditions, (i) and (ii), are satisfied. *Is the proof complete?*
No we have to go the other way, $\Leftarrow$, that is assume both conditions, (i) and (ii), are true then prove that the real number U is the supremum of the set S.
The given condition (i) says:
(i) For all $x \in S$ we have $x \le U$ which means that the real number U is an upper bound of the set S. *But how do we show that this real number, U, is the supremum, or the Least Upper Bound, of the set S?*
Again by contradiction. Suppose V is the supremum, LUB, of the set S where $V < U$. Then by condition (ii) which says:
(ii) For every $\varepsilon > 0$ there is a real number y in the set S, $y \in S$, such that

$$U-\varepsilon < y \le U$$

We can choose $\varepsilon > 0$ so that

$$V = U-\varepsilon < y$$

But $y \in S$ by condition (ii). This contradicts V is the supremum of the set S because $V < y$. Hence the real number U is the supremum of the set S.

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Proposition (5.6)
Let S be a non-empty subset of real numbers, $\mathbb{R}$. A real number L is the infimum of the set S if and only if it satisfies both the following conditions:
(i) For all $x \in S$ we have $L \le x$
(ii) For every $\varepsilon > 0$ there is a real number y in the set S, $y \in S$, such that

$$L \le y < L+\varepsilon$$

What does this proposition mean?
Assume that the real number L is the infimum of the set S then any number greater than L is **not** a lower bound of the set S. That is for every positive number ε, however small, the real number $L+\varepsilon$ is **not** a lower bound of the set S. This means that there is an element in the set S, say y, which is less than $L+\varepsilon$. Also if L is a lower bound and for every $\varepsilon > 0$ there is a real number $y \in S$ such that $L \le y < L+\varepsilon$ then L is the infimum of the set.

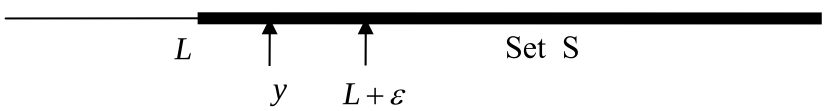

Fig 13
Proof. See Exercise 5b. Actually try proving it now because this will be good practice of understanding the above and remaining proofs.

Proposition (5.7)
A real number U of a non-empty subset S of real numbers, $\mathbb{R}$, is the supremum of S if and only if
(i) For all $x \in S$ we have $x \le U$
(ii) there exists a real number, v, such that $v < U$ then there is a $y \in S$ such that $v < y$.
What does this proposition, (5.7), mean?

It means that if the real number U is the supremum of a set S then for any real number less than U such as v we can find an element, y, in the set S such that $v < y$. Also if U is an upper bound and if $v < U$ and there is a $y \in S$ such that $v < y$ then the real number U is the supremum of the set.

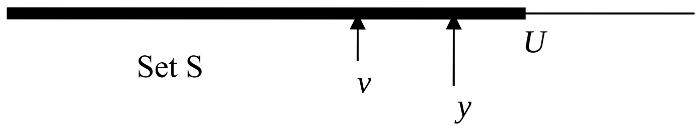

Fig 14

Proof. Applying the above proposition (5.5) by choosing $\varepsilon > 0$ such that the real number $v = U - \varepsilon$.

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B2 **Applications of Properties of Supremum and Infimum**

Note that for a non empty subset S of real numbers, $\mathbb{R}$, to have a supremum it **must** be bounded above. Conversely if the subset S is **not** bounded above then it **cannot** have a supremum. Additionally the supremum is real number, that is the supremum is in $\mathbb{R}$.

This does **not** apply to the set of rational numbers, $\mathbb{Q}$. For example consider the set

$$S = \left\{ x \,\middle|\, 0 \le x \le \sqrt{2} \text{ and } x \in \mathbb{Q} \right\}$$

Clearly the set S is bounded above by $\sqrt{2}$ but there is **no** rational number U which is the Least Upper Bound or supremum of the set S. Hence $\sup(S)$ does **not** exist in the rationals, $\mathbb{Q}$. This is the fundamental **difference** between the sets of rationals, $\mathbb{Q}$, and reals, $\mathbb{R}$.

So far all the non empty subsets of the reals, $\mathbb{R}$, which are bounded above have had a supremum, Least Upper Bound, in $\mathbb{R}$. This is a general result called the **Completeness Axiom**.

(5.8) Completeness Property of $\mathbb{R}$: Every non empty subset of the reals, $\mathbb{R}$,which is bounded above has a supremum, LUB, in $\mathbb{R}$.

From this we can deduce the analogous property for infimum of a non empty subset which is bounded below. See Exercise 5b.

Proposition (5.9)

Let $S = \{ x \mid a \le x \le b \text{ and } x \in \mathbb{R} \}$. Then $\sup(S) = b$ and $\inf(S) = a$.

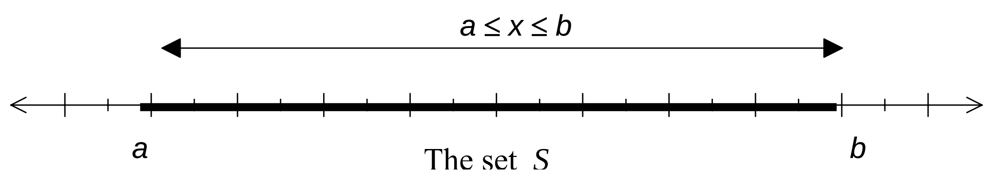

Fig 15

Proof. We only prove $\sup(S) = b$. You are asked to prove $\inf(S) = a$ in Exercise 5b.

How are we going to prove this, $\sup(S) = b$ *?*

We first assume $\sup(S) < b$ and then $\sup(S) > b$ and in both cases we arrive at a contradiction. Therefore $\sup(S) = b$. Here is the proof:

If $\sup(S) < b$ then this contradicts that $\sup(S)$ is an upper bound of the set S because $b \in S$ by the definition of the set $S = \{x \mid a \le x \le b \text{ and } x \in \mathbb{R}\}$.

If $b < \sup(S)$ then by the above proposition (5.7) we have a real number y in the set S, $y \in S$, such that

$$b < y$$

This contradicts the definition of the given set $S = \{x \mid a \le x \le b \text{ and } x \in \mathbb{R}\}$ where ALL the elements of the set are less than or equal to b. Hence $\sup(S) = b$.

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Proposition (5.10)

Let S and T be non-empty subsets of $\mathbb{R}$ that are bounded above by real numbers. Let the set $S+T$ be defined by

$$S+T = \{s+t \mid s \in S \text{ and } t \in T\}$$

Then $\sup(S+T) = \sup(S) + \sup(T)$.

Proof.

By the definition of supremum, (5.2), which means $\sup(S)$ and $\sup(T)$ are the Least Upper Bounds so we have for all $s \in S$ and for all $t \in T$

$$s \le \sup(S) \text{ and } t \le \sup(T)$$

$$s+t \le \sup(S) + \sup(T)$$

Therefore $\sup(S) + \sup(T)$ is an upper bound for the set $S+T$ which means

$$\underbrace{\sup(S+T)}_{\text{Least Upper Bound}} \le \underbrace{\sup(S)+\sup(T)}_{\text{Upper Bound}} \qquad (*)$$

Because $\sup(S+T)$ is the Least Upper Bound of the set $S+T$.

What do we need to show?

Required to show that the Left Hand Side is equal to the Right Hand Side of (*):

$$\sup(S+T) = \sup(S) + \sup(T)$$

How do we show this result?

By contradiction. Suppose

$$\sup(S+T) < \sup(S) + \sup(T) \qquad \text{[LHS is Strictly Less Than RHS of (*)]}$$

Then

$$\sup(S+T) - \sup(T) < \sup(S)$$

By the above proposition (5.7) there is a real number y in the set S, $y \in S$, such that

$$\sup(S+T) - \sup(T) < y$$

$$\sup(S+T) - y < \sup(T) \qquad \text{[Rearranging]}$$

Again by proposition (5.7) there is a real number x in the set T, $x \in T$, such that

$$\sup(S+T) - y < x$$

$$\sup(S+T) < x + y \qquad \text{[Rearranging]}$$

which is impossible because $\sup(S+T)$ is an upper bound for the set $S+T$ and the real number $(x+y) \in S+T$.

We contradict our supposition that $\sup(S+T)<\sup(S)+\sup(T)$. Hence we have the required result, $\sup(S+T)=\sup(S)+\sup(T)$.

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SUMMARY

Proposition (5.4)

(i) The supremum of a set is unique.

(ii) The infimum of a set is unique.

Proposition (5.7)

An upper bound U of a non-empty subset S of $\mathbb{R}$ is the supremum of S if and only if $v<U$ then there is a $y \in S$ such that $v<y$.

(5.8) Completeness Property of $\mathbb{R}$: Every non empty subset of the reals, $\mathbb{R}$,which is bounded above has a supremum, LUB, in $\mathbb{R}$.