1. a) $\sum_{n=1}^{\infty}\sqrt{n}$ b) $\sum_{n=1}^{\infty}(2n)$ c) $\sum_{n=1}^{\infty}(2n-1)$

d) $\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{n}\right]$ e) $\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n$ f) $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$

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2. Each of these is a geometric series with the common ratio *r* less than 1 so we can use the following formula to find the sum:

(7.27) $$\sum_{n=1}^{\infty}ar^{n-1}=\frac{a}{1-r}=\frac{\text{First term}}{1-\text{Common ratio}}$$

a) For the series $\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ the first term is $a=\frac{1}{3}$ because we start with index 1, that is $\left(\frac{1}{3}\right)^1=\frac{1}{3}$. Common ratio $r=\frac{1}{3}$. Substituting these into the above formula (7.27) gives

$$\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n=\frac{1/3}{1-1/3}=\frac{1/3}{2/3}=\frac{1}{2}\qquad\left[\begin{array}{l}\text{Multiplying numerator and}\\ \text{denominator by 3}\end{array}\right]$$

b) Similarly for $\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n$ we have $a=r=\frac{1}{4}$. Substituting these into the above formula

(7.27) $$\sum_{n=1}^{\infty}ar^{n-1}=\frac{a}{1-r}=\frac{\text{First term}}{1-\text{Common ratio}}$$

gives

$$\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n=\frac{1/4}{1-1/4}=\frac{1/4}{3/4}=\frac{1}{3}\qquad\left[\begin{array}{l}\text{Multiplying numerator and}\\ \text{denominator by 4}\end{array}\right]$$

c) Very similar to parts (a) and (b), we have

$$\sum_{n=1}^{\infty}\left(\frac{1}{\pi}\right)^n=\frac{\pi}{1-1/\pi}$$

$$=\frac{\pi}{(\pi-1)/\pi}=\frac{1}{\pi-1}\qquad\left[\begin{array}{l}\text{Multiplying numerator and}\\ \text{denominator by }\pi\end{array}\right]$$

d) This time we have $m>1$ therefore the common ratio $\frac{1}{m}<1$ so we can apply the above formula to find the sum of the infinite series:

$$\sum_{n=1}^{\infty}\left(\frac{1}{m}\right)^n=\frac{1/m}{1-\frac{1}{m}}$$

$$=\frac{1/m}{(m-1)/m}=\frac{1}{m-1}\qquad\left[\begin{array}{l}\text{Multiplying numerator and}\\ \text{denominator by }m\end{array}\right]$$

3. a) We have

$$\sum_{n=1}^{\infty}\left(\frac{1}{2^{2n-1}}\right)=\frac{1}{2^{(2\times1)-1}}+\frac{1}{2^{(2\times2)-1}}+\frac{1}{2^{(2\times3)-1}}+\frac{1}{2^{(2\times4)-1}}+\cdots$$

$$=\frac{1}{2^1}+\frac{1}{2^3}+\frac{1}{2^5}+\frac{1}{2^7}+\cdots$$

What is the common ratio in this case?

Common ratio $r=\frac{1}{2^2}=\frac{1}{4}$ which is less than 1 so the series converges and we can use the formula

(7.27) $$\sum_{n=1}^{\infty}ar^{n-1}=\frac{a}{1-r}=\frac{\text{First term}}{1-\text{Common ratio}}$$

What is the first term a equal to?

$a=\frac{1}{2}$. Substituting $a=\frac{1}{2}$ and $r=\frac{1}{4}$ into the above formula (7.27) we have

$$\sum_{n=1}^{\infty}ar^{n-1}=\frac{a}{1-r}=\frac{1/2}{1-1/4}$$

$$=\frac{1/2}{3/4}=\frac{2}{3}\qquad\left[\begin{array}{l}\text{Multiplying numerator and}\\ \text{denominator by 4}\end{array}\right]$$

b) Diverges because we are given $\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n$ which is

$$\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n=\frac{3}{2}+\left(\frac{3}{2}\right)^2+\left(\frac{3}{2}\right)^3+\left(\frac{3}{2}\right)^4+\cdots$$

What is the common ratio r equal to?

$$r=\frac{3}{2}\quad\text{which is greater than 1}$$

Hence the series diverges.

c) Diverges because we have $\sum_{n=1}^{\infty}(e)^n$ and the common ratio

$$r=e=2.71828\cdots$$

The common ratio is greater than 1 so the series diverges.

d) *Does the given series* $\sum_{n=1}^{\infty}10\left(\frac{1}{3}\right)^n$ *converge or not?*

Writing out this series we have

$$\sum_{n=1}^{\infty}10\left(\frac{1}{3}\right)^n=10\left(\frac{1}{3}\right)^1+10\left(\frac{1}{3}\right)^2+10\left(\frac{1}{3}\right)^3+10\left(\frac{1}{3}\right)^4+\cdots$$

$$\underbrace{=}_{\substack{\text{Taking out a}\\ \text{common factor of 10}}}10\left[\left(\frac{1}{3}\right)+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^3+\left(\frac{1}{3}\right)^4+\cdots\right]$$

The common ratio is $\frac{1}{3}$ which is less than 1 so the given series converges and

$$\sum_{n=1}^{\infty} 10\left(\frac{1}{3}\right)^n = \frac{10/3}{1-1/3}$$
$$= \frac{10/3}{2/3} = \frac{10}{2} = 5$$

4. The total distance D travelled by the ball is given by the infinite series:

$$D = 10 + \underbrace{(10\times 0.55)}_{\text{After first bounce}} + \underbrace{(10\times 0.55^2)}_{\text{After second bounce}} + \underbrace{(10\times 0.55^3)}_{\text{After third bounce}} + \cdots$$
$$= \sum_{n=0}^{\infty} 10(0.55)^n$$

How can we find D?

D is a geometric series with first term $a = 10$ and common ratio $r = 0.55$. Since $|r| = 0.55$ is less than 1 therefore the sum of this series is given by

$$D = \frac{10}{1-0.55} = 22.22\text{ m} \qquad \left[\frac{\text{First term}}{1-(\text{Common ratio})}\right]$$

Hence the total distance travelled by the ball is 22.22m (2dp).

5. The maximum rise R of the balloon is given by:

$$R = 50 + \underbrace{(50\times 0.65)}_{\text{After second minute}} + \underbrace{(50\times 0.65^2)}_{\text{After third minute}} + \underbrace{(50\times 0.65^3)}_{\text{After fourth minute}} + \cdots = \sum_{n=0}^{\infty} 50(0.65)^n$$

How can we find R?

R is a geometric series with first term $a = 50$ and common ratio $r = 0.65$. Since $|r| = 0.65$ is less than 1 therefore the sum of this series is given by

$$D = \frac{50}{1-0.65} = 142.86\text{ m} \qquad \left[\frac{\text{First term}}{1-(\text{Common ratio})}\right]$$

Hence the maximum rise by the balloon is 142.86m (2dp).

6. We are given that the area removed is $A = \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^n$. Writing this out we have:

$$A = \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^n = \frac{1}{4} + \frac{1}{4}\left(\frac{3}{4}\right) + \frac{1}{4}\left(\frac{3}{4}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^2 + \cdots$$

How can we find A?

A is a geometric series with first term $a = \frac{1}{4}$ and common ratio $r = \frac{3}{4}$. Since $|r|$ is less than 1 therefore we can find the sum of this infinite series. We have

$$A = \frac{1/4}{1-3/4} = 1 \qquad \left[\frac{\text{First term}}{1-(\text{Common ratio})}\right]$$

$A = 1$ means that the whole area is removed.

7. (a) *Is* $S = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$ *a geometric series?*

Yes because each of term is 1/10 of the previous term. *What is the sum of this series?*
Since the common ratio $r = \frac{1}{10}$ which means that the modulus of this is less than 1 therefore the sum of the given infinite series is

$$S = \frac{9/10}{1-1/10} = 1 \qquad \left[\frac{\text{First term}}{1-(\text{Common ratio})}\right]$$

This means that $S = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots = 0.999\cdots = 1$.

(b) Similarly we are given that $S = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots$ which is a geometric series with the same common ratio of $r = \frac{1}{10}$ and first term $a = \frac{9}{10}$ which means we can find the sum of this infinite series.

$$S = \frac{3/10}{1-1/10} = \frac{1}{3}$$

As part (a) this means that $S = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots = 0.333\cdots = \frac{1}{3}$.

(c) A carbon copy of the solutions presented in parts (a) and (b) gives that the sum of

$$S = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots$$

with $a = 1/10, \quad r = 1/10$ is

$$S = \frac{1/10}{1-1/10} = \frac{1}{9}$$

We conclude that this means $S = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots = 0.111\cdots = \frac{1}{9}$.

Parts (a), (b) and (c) show that $0.999\cdots = 1$, $0.333\cdots = \frac{1}{3}$ and $0.111\cdots = \frac{1}{9}$.

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8. The total profit P is given by

$$P = 100 + 0.91(100) + 0.91^2(100) + 0.91^3(100) + \cdots$$

This is a geometric series with first term $a = 100$ and common ratio $r = 0.91$. Since the common ratio is $|r| = |0.91| = 0.91 < 1$ therefore the series converges. We have

$$P = \frac{100}{1-0.91} = 1111.11 \qquad \left[\frac{\text{First term}}{1-(\text{Common ratio})}\right]$$

The total possible profit is £1111.11 (2dp).

---

9. (a) We need to test the given series $8+4+2+1+\ldots$ for convergence. *How can we write this series in compact form?*

$$8+4+2+1+\ldots=8+\frac{1}{2}8+\left(\frac{1}{2}\right)^2 8+\left(\frac{1}{2}\right)^3 8+\cdots$$
$$=8\left(1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\cdots\right)$$
$$=8\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$$

The series $8\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$ is a geometric series. *Does this series converge?*

The common ratio is $\frac{1}{2}$ so the series converges and we use the formula

(7.27) $$\sum_{n=1}^{\infty} ar^{n-1}=\frac{a}{1-r}=\frac{\text{First term}}{1-\text{Common ratio}}$$

to find the sum of the infinite series. *What is the first term in this case?*

Clearly it is 8. Hence substituting $a=8$ and $r=\frac{1}{2}$ into (7.27) gives

$$8\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n=\frac{8}{1-1/2}=\frac{8}{1/2}=16$$

The sum of the infinite series is 16.

(b) The given series $3+6+12+24+\ldots$ diverges. *Why?*

By (7.25) we have $\lim_{n\to\infty}(a_n)\neq 0$ then $\sum(a_n)$ diverges. This means that if the nth term does **not** tend towards zero then the series diverges. Since our series $3+6+12+24+\ldots$ gets bigger so it diverges.

(c) For the given series $16+12+9+\frac{27}{4}+\ldots$ it is difficult to write down a formula in compact form. However we can divide two consecutive terms:

$$\frac{12}{16}=\frac{9}{12}=\frac{27/4}{9}=\cdots=\frac{3}{4}$$

This means we have a geometric series with a common ratio of ¾. The first term is 16 and so the sum of the infinite series is

$$16+12+9+\frac{27}{4}+\ldots=\frac{\text{First term}}{1-\text{Common ratio}}=\frac{16}{1-3/4}=\frac{16}{1/4}=64$$

10. (a) We are given the series:

$$\sum_{n=1}^{\infty}\frac{1}{x^n}=\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\frac{1}{x^4}+\cdots$$

What is the common ratio r equal to?

$r=\frac{1}{x}$. Since $|x|>1$ which means that $|r|=\left|\frac{1}{x}\right|<1$ so the series converges and the sum is

$$\sum_{n=1}^{\infty}\frac{1}{x^n}=\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\frac{1}{x^4}+\cdots$$

$$=\frac{\text{First term}}{1-\text{Common ratio}}$$

$$=\frac{1/x}{1-\frac{1}{x}}=\frac{1}{x-1}\quad\begin{bmatrix}\text{Multiplying numerator}\\ \text{and denominator by } x\end{bmatrix}$$

(b) We are given the series $\sum_{n=1}^{\infty}\frac{x^n}{2^n}=\sum_{n=1}^{\infty}\left(\frac{x}{2}\right)^n=\frac{x}{2}+\left(\frac{x}{2}\right)^2+\left(\frac{x}{2}\right)^3+\left(\frac{x}{2}\right)^4+\cdots$

We have $|x|<2$ therefore $|r|=\left|\frac{x}{2}\right|<1$ which means that the series converges. Using

(7.27) $$\sum_{n=1}^{\infty}ar^{n-1}=\frac{a}{1-r}=\frac{\text{First term}}{1-\text{Common ratio}}$$

we have

$$\sum_{n=1}^{\infty}\left(\frac{x}{2}\right)^n=\frac{\text{First term}}{1-\text{Common ratio}}$$

$$=\frac{x/2}{1-\frac{x}{2}}=\frac{x}{2-x}\quad\begin{bmatrix}\text{Multiplying numerator}\\ \text{and denominator by } 2\end{bmatrix}$$

(c) Similarly we have

$$\sum_{n=1}^{\infty}\frac{1}{(1+x)^n}=\frac{1}{1+x}+\frac{1}{(1+x)^2}+\frac{1}{(1+x)^3}+\frac{1}{(1+x)^4}+\cdots$$

What is the common ratio r and first term a equal to in this case?

$$r=a=\frac{1}{1+x}$$

Since we are given that $x>0$ so $r=\frac{1}{1+x}<1$. Hence the series converges and

$$\sum_{n=1}^{\infty}\frac{1}{(1+x)^n}=\frac{\text{First term}}{1-\text{Common ratio}}$$

$$=\frac{1/(x+1)}{1-\frac{1}{x+1}}=\frac{1}{x}\quad\begin{bmatrix}\text{Multiplying numerator}\\ \text{and denominator by } x+1\end{bmatrix}$$

(d) The series given in this part is very similar to the one in part (c) above. We have

$$\sum_{n=1}^{\infty}\frac{1}{(1+x^2)^n}=\frac{1}{1+x^2}+\frac{1}{(1+x^2)^2}+\frac{1}{(1+x^2)^3}+\frac{1}{(1+x^2)^4}+\cdots$$

Also $r=a=\frac{1}{1+x^2}$. We are give that $x\neq 0$ so $r=\frac{1}{1+x^2}<1$. Hence

$$\sum_{n=1}^{\infty}\frac{1}{\left(1+x^2\right)^n}=\frac{a}{1-r}$$

$$=\frac{1/\left(1+x^2\right)}{1-\dfrac{1}{1+x^2}}=\frac{1}{x^2} \qquad \begin{bmatrix}\text{Multiplying numerator} \\ \text{and denominator by } 1+x^2\end{bmatrix}$$

11. In many of these cases we apply the ratio test which is $\lim_{n\to\infty}\left(\dfrac{a_{n+1}}{a_n}\right)=L$. The series only converges if $L<1$.

(a) We are given $\sum\left(\dfrac{1}{(2n)!}\right)$. This means that $a_n=\dfrac{1}{(2n)!}$ and so the next term $n+1$ is

$$a_{n+1}=\frac{1}{\left(2(n+1)\right)!}=\frac{1}{(2n+2)!}$$

Substituting these $a_n=\dfrac{1}{(2n)!}$ and $a_{n+1}=\dfrac{1}{(2n+2)!}$ into $L=\lim_{n\to\infty}\left(\dfrac{a_{n+1}}{a_n}\right)$ gives

$$L=\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)$$

$$=\lim_{n\to\infty}\left[\frac{1}{(2n+2)!}\div\frac{1}{(2n)!}\right]$$

$$=\lim_{n\to\infty}\left[\frac{1}{(2n+2)!}\times\frac{(2n)!}{1}\right] \qquad \begin{bmatrix}\text{Inverting the second} \\ \text{fraction and multiplying}\end{bmatrix}$$

$$=\lim_{n\to\infty}\left[\frac{(2n)!}{(2n+2)!}\right]=\lim_{n\to\infty}\left[\frac{1}{(2n+2)(2n+1)}\right]=0$$

Since $L=0$ the series converges.

(b) We need to evaluate L to test for convergence. For $\sum_{n=0}^{\infty}\left(\dfrac{n!}{2^n}\right)$ we have $a_n=\dfrac{n!}{2^n}$ and so $a_{n+1}=\dfrac{(n+1)!}{2^{n+1}}$. Substituting these into $L=\lim_{n\to\infty}\left(\dfrac{a_{n+1}}{a_n}\right)$ gives

$$L=\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)$$

$$=\lim_{n\to\infty}\left(\frac{(n+1)!}{2^{n+1}}\div\frac{n!}{2^n}\right)$$

$$=\lim_{n\to\infty}\left(\frac{(n+1)!}{2^{n+1}}\times\frac{2^n}{n!}\right)=\lim_{n\to\infty}\left(\frac{n+1}{2}\right)=\infty$$

Since $L=+\infty$ the given series diverges.

(c) Very similar to part (b) with the 2 being replaced by 3. We find that $L=+\infty$ so the given series diverges.

(d) We have the series $\sum\left(\frac{(n+1)^2}{2^n}\right)$ and we need to determine $L=\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)$. In this case

$a_n=\frac{(n+1)^2}{2^n}$ and replacing n with $n+1$ yields $a_{n+1}=\frac{(n+2)^2}{2^{n+1}}$

$$
\begin{aligned}
L&=\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)\\
&=\lim_{n\to\infty}\left[\frac{(n+2)^2}{2^{n+1}}\div\frac{(n+1)^2}{2^n}\right]\\
&=\lim_{n\to\infty}\left[\frac{(n+2)^2}{2^{n+1}}\times\frac{2^n}{(n+1)^2}\right] && \left[\begin{array}{l}\text{Inverting the second}\\ \text{fraction and multiplying}\end{array}\right]\\
&=\lim_{n\to\infty}\left[\frac{1}{2}\left(\frac{n+2}{n+1}\right)^2\right]\\
&=\lim_{n\to\infty}\left[\frac{1}{2}\left(\frac{n+1+1}{n+1}\right)^2\right]=\lim_{n\to\infty}\left[\frac{1}{2}\left(\frac{n+1}{n+1}+\frac{1}{n+1}\right)^2\right]=\lim_{n\to\infty}\left[\frac{1}{2}\left(1+\frac{1}{n+1}\right)^2\right]=\frac{1}{2}(1+0)^2=\frac{1}{2}
\end{aligned}
$$

Since $L=\frac{1}{2}<1$ the series converges.

(e) We are given the series $\sum\left(e^{-n}\right)$ which can be in written in expanded form as

$$
\begin{aligned}
\sum\left(e^{-n}\right)&=e^{-1}+e^{-2}+e^{-3}+e^{-4}+\cdots\\
&=\frac{1}{e}+\frac{1}{e^2}+\frac{1}{e^3}+\frac{1}{e^4}+\cdots
\end{aligned}
$$

This is a geometric series with $r=a=\frac{1}{e}=\frac{1}{2.71828\cdots}$. Since $r<1$ so the series converges.

By using the ratio test we get $L=\frac{1}{e}$. (The sum is $\frac{1}{e-1}$.)

(f) We have $\sum\left(\frac{n^2}{3^n}\right)$. To use the ratio test we need to find a_n and a_{n+1}. *What are these equal to?*

$a_n=\frac{n^2}{3^n}$ and $a_{n+1}=\frac{(n+1)^2}{3^{n+1}}$. Putting these into $L=\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)$ gives

$$L = \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)$$

$$= \lim_{n\to\infty}\left[\frac{(n+1)^2}{3^{n+1}} \div \frac{n^2}{3^n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{(n+1)^2}{3^{n+1}} \times \frac{3^n}{n^2}\right] \qquad \left[\begin{array}{l}\text{Turning the second}\\ \text{fraction upside down}\end{array}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{3}\frac{(n+1)^2}{n^2}\right] = \lim_{n\to\infty}\left[\frac{1}{3}\left(\frac{n+1}{n}\right)^2\right] \underbrace{=}_{\substack{\text{Dividing numerator}\\ \text{and denominator by } n}} \lim_{n\to\infty}\left[\frac{1}{3}\left(\frac{1+1/n}{1}\right)^2\right] = \frac{1}{3}$$

Since $L = 1/3$ which is less than 1 so the series converges.

(g) We need to test the series $\sum\left(\frac{10^n}{n!}\right)$ for convergence. *How?*

By using the ratio test. Let $a_n = \frac{10^n}{n!}$ then $a_{n+1} = \frac{10^{n+1}}{(n+1)!}$. We have

$$L = \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)$$

$$= \lim_{n\to\infty}\left[\frac{10^{n+1}}{(n+1)!} \div \frac{10^n}{n!}\right]$$

$$= \lim_{n\to\infty}\left[\frac{10^{n+1}}{(n+1)!} \times \frac{n!}{10^n}\right] = \lim_{n\to\infty}\left[\frac{10}{n+1}\right] = 0$$

Since L is less than 1 so the given series converges.

(h) Similarly for $\sum\left(\frac{3^n n}{(n+1)^2}\right)$ we use the ratio test. In this case $a_n = \frac{3^n n}{(n+1)^2}$ and $a_{n+1} = \frac{3^{n+1}(n+1)}{(n+2)^2}$. Putting these into $L = \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)$ gives

$$L=\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)$$

$$=\lim_{n\to\infty}\left(\frac{3^{n+1}(n+1)}{(n+2)^2}\div\frac{3^n n}{(n+1)^2}\right)$$

$$=\lim_{n\to\infty}\left(\frac{3^{n+1}(n+1)}{(n+2)^2}\times\frac{(n+1)^2}{3^n n}\right)$$

$$=\lim_{n\to\infty}\left(\frac{3(n+1)}{n}\times\frac{(n+1)^2}{(n+2)^2}\right)$$

$$=\lim_{n\to\infty}\left(3\frac{(n+1)}{n}\times\left(\frac{n+1}{n+2}\right)^2\right)$$

$$\underbrace{=}_{\substack{\text{Dividing numerator}\\ \text{and denominator by } n}}\lim_{n\to\infty}\left(3\frac{(1+1/n)}{1}\times\left(\frac{1+1/n}{1+2/n}\right)^2\right)=\lim_{n\to\infty}\left(3(1)\times(1)^2\right)=3$$

We have $L=3$ therefore the series diverges.

(i) We are given $\sum\left(\frac{n!}{(2n+1)!}\right)$. We have $a_n=\frac{n!}{(2n+1)!}$ therefore

$$a_{n+1}=\frac{(n+1)!}{(2(n+1)+1)!}=\frac{(n+1)!}{(2n+3)!}$$

Evaluating L we have

$$L=\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)$$

$$=\lim_{n\to\infty}\left(\frac{(n+1)!}{(2n+3)!}\div\frac{n!}{(2n+1)!}\right)$$

$$=\lim_{n\to\infty}\left(\frac{(n+1)!}{(2n+3)!}\times\frac{(2n+1)!}{n!}\right)$$

$$=\lim_{n\to\infty}\left(\frac{n+1}{(2n+3)(2n+2)}\right)$$

$$=\lim_{n\to\infty}\left(\frac{n+1}{4n^2+10n+6}\right)\underbrace{=}_{\substack{\text{Dividing numerator}\\ \text{and denominator by } n}}\lim_{n\to\infty}\left(\frac{1+1/n}{4n+10+6/n}\right)=0$$

Since L is equal to zero so the series converges.

(j) We need to test $\sum\left(\frac{11^n}{2^{n+1}n}\right)$ for convergence. Let $a_n=\frac{11^n}{2^{n+1}n}$ then $a_{n+1}=\frac{11^{n+1}}{2^{n+2}(n+1)}$:

$$L=\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)$$

$$=\lim_{n\to\infty}\left(\frac{11^{n+1}}{2^{n+2}(n+1)}\div\frac{11^n}{2^{n+1}n}\right)$$

$$=\lim_{n\to\infty}\left(\frac{11^{n+1}}{2^{n+2}(n+1)}\times\frac{2^{n+1}n}{11^n}\right)$$

$$=\lim_{n\to\infty}\left(\frac{11}{2}\frac{n}{n+1}\right)\underbrace{=}_{\substack{\text{Dividing numerator}\\ \text{and denominator by } n}}\lim_{n\to\infty}\left(\frac{11}{2}\frac{1}{1+1/n}\right)=\frac{11}{2}$$

We have $L=\frac{11}{2}>1$ therefore the given series diverges.

---

12. In each case we show that $L=\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)$ is equal to 1.

(a) We are given the series $\sum\left(\frac{1}{n^3}\right)$ which means that $a_n=\frac{1}{n^3}$ and $a_{n+1}=\frac{1}{(n+1)^3}$:

$$L=\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)$$

$$=\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)$$

$$=\lim_{n\to\infty}\left[\frac{1}{(n+1)^3}\times n^3\right]$$

$$=\lim_{n\to\infty}\left[\left(\frac{n}{n+1}\right)^3\right]\underbrace{=}_{\substack{\text{Dividing numerator}\\ \text{and denominator by } n}}\lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)=\frac{1}{1+0}=1$$

Hence the ratio test fails for this series.

(b) Similarly for $\sum\left(\frac{1}{n+10}\right)$ we find that $L=1$.

(c) For the given series $\sum\left(\frac{1}{n^2+1}\right)$ we have $a_n=\frac{1}{n^2+1}$ therefore $a_{n+1}=\frac{1}{(n+1)^2+1}$:

$$L=\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)$$
$$=\lim_{n\to\infty}\left(\frac{1}{(n+1)^2+1}\div\left(\frac{1}{n^2+1}\right)\right)$$
$$=\lim_{n\to\infty}\left(\frac{1}{(n+1)^2+1}\times\left(\frac{n^2+1}{1}\right)\right)$$
$$=\lim_{n\to\infty}\left(\frac{n^2+1}{n^2+2n+1+1}\right)$$
$$=\lim_{n\to\infty}\left(\frac{n^2+1}{n^2+2n+2}\right)\underbrace{=}_{\substack{\text{Dividing numerator}\\ \text{and denominator by } n^2}}\lim_{n\to\infty}\left(\frac{1+1/n^2}{1+2/n+2/n^2}\right)=\frac{1+0}{1+0+0}=1$$

Since $L=1$ the ratio test fails.

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13. (a) **i** We are given $\sum\left(\frac{2^n n!}{n^n}\right)$. Let $a_n=\frac{2^n n!}{n^n}$ therefore $a_{n+1}=\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}$. We have

$$L=\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)$$
$$=\lim_{n\to\infty}\left(\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}\div\frac{2^n n!}{n^n}\right)$$
$$=\lim_{n\to\infty}\left(\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}\times\frac{n^n}{2^n n!}\right)$$
$$=\lim_{n\to\infty}\left(2\frac{n+1}{n+1}\left(\frac{n}{n+1}\right)^n\right)$$
$$=\lim_{n\to\infty}\left(2\left(\frac{n}{n+1}\right)^n\right)=2\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\frac{2}{e}$$

Since $L=\frac{2}{e}<1$ the given series converges.

ii Very similar to part **i**. We get $L=\frac{3}{e}>1$ so the series diverges.

(b) We have $\sum\left(\frac{x^n n!}{n^n}\right)$. Let $a_n=\frac{x^n n!}{n^n}$ therefore $a_{n+1}=\frac{x^{n+1}(n+1)!}{(n+1)^{n+1}}$. Determining L:

$$
\begin{aligned}
L &= \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) \\
&= \lim_{n\to\infty}\left(\frac{x^{n+1}(n+1)!}{(n+1)^{n+1}} \div \frac{x^n n!}{n^n}\right) \\
&= \lim_{n\to\infty}\left(\frac{x^{n+1}(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{x^n n!}\right) \\
&= \lim_{n\to\infty}\left(x\frac{n+1}{n+1}\left(\frac{n}{n+1}\right)^n\right) \\
&= \lim_{n\to\infty}\left(x\left(\frac{n}{n+1}\right)^n\right) = x\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{x}{e}
\end{aligned}
$$

Remember the series converges if L is less than 1. We have

(i) $0 < x < e$ (ii) $x > e$