Complete Solutions to Exercise 1g

2. *Proof*. Let $P(n)$ be the given proposition: $2+5+8+...+(3n-1)=\frac{1}{2}n(3n+1)$

Check $P(1)$. Substituting $n=1$ gives

$$2=\frac{1}{2}(1)(3+1) \qquad \surd$$

Hence $P(1)$ is true. Assume the proposition is true for $n=k$:

$$2+5+8+...+(3k-1)=\frac{1}{2}k(3k+1) \qquad (*)$$

Required to prove the result for $n=k+1$. We need to prove

$$\begin{aligned}2+5+8+...+(3k-1)+(3(k+1)-1)&=\frac{1}{2}(k+1)(3(k+1)+1)\\&=\frac{1}{2}(k+1)(3k+4) \qquad (**)\end{aligned}$$

How do we prove (**)?
By examining the Left Hand Side and using (*).

$$\begin{aligned}2+5+...+(3k-1)+(3(k+1)-1)&=\underbrace{2+5+8+...+(3k-1)}_{=\frac{1}{2}k(3k+1)\text{ by }(*)}+\underbrace{(3(k+1)-1)}_{=3k+2}\\&=\frac{1}{2}k(3k+1)+(3k+2)\\&=\frac{1}{2}\left[k(3k+1)+2(3k+2)\right] \qquad \left[\text{Rewriting } (3k+2)=\frac{1}{2}2(3k+2)\right]\\&=\frac{1}{2}\left[3k^2+\underbrace{k+6k}_{=7k}+4\right] \qquad \left[\text{Expanding Brackets}\right]\\&=\frac{1}{2}\left[3k^2+7k+4\right]\\&=\frac{1}{2}\left[(k+1)(3k+4)\right] \qquad \left[\text{Factorizing Quadratic}\right]\end{aligned}$$

The last line is the Right Hand Side of (**). Therefore we have shown (**) and by induction we have our given proposition.

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4. *Proof*. Let $P(n)$ be the given proposition: $1^3+2^3+3^3+...+n^3=(1+2+3+4+...+n)^2$

Check $P(1)$. Substituting $n=1$ gives

$$1^3=(1)^2 \qquad \surd$$

Hence $P(1)$ is true. Assume the proposition is true for $n=k$:

$$1^3+2^3+3^3+...+k^3=(1+2+3+4+...+k)^2$$

Required to prove the proposition for $n=k+1$:

$$1^3+2^3+3^3+...+k^3+(k+1)^3=(1+2+3+4+...+k+(k+1))^2 \qquad (\dagger)$$

Using the given hint on the Left Hand Side of (†) gives

$$1^3+2^3+3^3+\ldots+k^3+(k+1)^3=\frac{1}{4}(k+1)^2(k+2)^2 \qquad (††)$$

[By Question 3 with $n=k+1$]

How do we show this is equal to the Right Hand Side of (†)?
By Example 43 which is

$$1+2+3+4+\ldots+n=\frac{1}{2}n(n+1)$$

Substituting $n=k+1$ into this we have

$$1+2+3+4+\ldots+(k+1)=\frac{1}{2}(k+1)(k+2)$$

Squaring both sides gives

$$\begin{aligned}(1+2+3+4+\ldots+(k+1))^2&=\left[\frac{1}{2}(k+1)(k+2)\right]^2\\&=\frac{1}{4}(k+1)^2(k+2)^2\end{aligned}$$

This the same as the Right Hand Side of (††). Therefore we have shown (†) which means the result follows by induction.

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10. *Proof*. Let $P(n)$ be the given proposition:

$$1^4+2^4+3^4+\ldots+n^4=\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

Check $P(1)$. Substituting $n=1$ gives

$$1^4=\frac{1(1+1)(2+1)(3+3-1)}{30}=\frac{1(2)(3)(5)}{30}=\frac{30}{30}=1 \quad \surd$$

Hence $P(1)$ is true. Assume the proposition is true for $n=k$:

$$1^4+2^4+3^4+\ldots+k^4=\frac{k(k+1)(2k+1)(3k^2+3k-1)}{30} \qquad (*)$$

Required to prove the proposition for $n=k+1$:

$$\begin{aligned}1^4+2^4+3^4+\ldots+k^4+(k+1)^4&=\frac{(k+1)((k+1)+1)(2(k+1)+1)(3(k+1)^2+3(k+1)-1)}{30}\\&=\frac{(k+1)(k+2)(2k+3)(3(k^2+2k+1)+3k+3-1)}{30} \quad \left[\begin{array}{l}\text{Simplifying}\\ \text{and Expanding}\end{array}\right]\\&=\frac{(k+1)(k+2)(2k+3)(3k^2+6k+3+3k+2)}{30}\\&=\frac{(k+1)(k+2)(2k+3)(3k^2+9k+5)}{30} \qquad (**)\end{aligned}$$

Expanding the Left Hand Side of (**) using (*) gives

$$1^4+2^4+3^4+\ldots+k^4+(k+1)^4=\underbrace{1^4+2^4+3^4+\ldots+k^4}_{=\frac{k(k+1)(2k+1)(3k^2+3k-1)}{30}\text{ by }(*)}+(k+1)^4$$

$$=\frac{k(k+1)(2k+1)(3k^2+3k-1)}{30}+(k+1)^4$$

$$=\frac{(k+1)}{30}\left[k(2k+1)(3k^2+3k-1)+30(k+1)^3\right]$$

Expanding the square brackets gives:

$$\left[k(2k+1)(3k^2+3k-1)+30(k+1)^3\right]=(2k^2+k)(3k^2+3k-1)+30(k^3+3k^2+3k+1)$$

$$=6k^4+6k^3-2k^2+3k^3+3k^2-k+30k^3+90k^2+90k+30$$

$$=6k^4+39k^3+91k^2+89k+30$$

Left Hand Side of (**) is equal to

$$\frac{(k+1)}{30}\left[k(2k+1)(3k^2+3k-1)+30(k+1)^3\right]=\frac{(k+1)}{30}\left[6k^4+39k^3+91k^2+89k+30\right]$$

Expanding the Right Hand Side of (**) also gives this result:

$$\frac{(k+1)(k+2)(2k+3)(3k^2+9k+5)}{30}=\frac{(k+1)}{30}\underbrace{\left[(k+2)(2k+3)(3k^2+9k+5)\right]}_{=6k^4+39k^3+91k^2+89k+30}$$

$$=\frac{(k+1)}{30}\left[6k^4+39k^3+91k^2+89k+30\right]$$

Hence the Left Hand Side is equal to the Right Hand Side of (**). We have shown $P(k) \Rightarrow P(k+1)$ therefore our given result follows by induction,

$$1^4+2^4+3^4+\ldots+n^4=\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

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11. *Proof*. Let $P(n)$ be the given proposition:

$$1^5+2^5+3^5+\ldots+n^5=\frac{n^2(n+1)^2(2n^2+2n-1)}{12}$$

Check $P(1)$. Substituting $n=1$ gives

$$1^5=\frac{1^2(1+1)^2\left(2(1)^2+2(1)-1\right)}{12}=\frac{2^2(2+2-1)}{12}=\frac{4(3)}{12}=1 \quad \surd$$

Hence $P(1)$ is true. Assume the proposition is true for $n=k$:

$$1^5+2^5+3^5+\ldots+k^5=\frac{k^2(k+1)^2(2k^2+2k-1)}{12} \qquad (€)$$

Required to prove the proposition for $n=k+1$:

$$1^5+2^5+3^5+\ldots+k^5+(k+1)^5=\frac{(k+1)^2\left((k+1)+1\right)^2\left(2(k+1)^2+2(k+1)-1\right)}{12}$$

$$=\frac{(k+1)^2(k+2)^2\left(2\left(k^2+2k+1\right)+2k+2-1\right)}{12}$$

$$=\frac{(k+1)^2(k+2)^2\left(2k^2+4k+2+2k+2-1\right)}{12}$$

$$=\frac{(k+1)^2(k+2)^2\left(2k^2+6k+3\right)}{12} \qquad (!)$$

Expanding the Left Hand Side of (!) using (€) gives

$$1^5+2^5+3^5+\ldots+k^5+(k+1)^5=\underbrace{1^5+2^5+3^5+\ldots+k^5}_{=\frac{k^2(k+1)^2\left(2k^2+2k-1\right)}{12}}+(k+1)^5$$

$$=\frac{k^2(k+1)^2\left(2k^2+2k-1\right)}{12}+(k+1)^5$$

$$=\frac{(k+1)^2}{12}\left[k^2\left(2k^2+2k-1\right)+12(k+1)^3\right] \qquad \left[\text{Taking Out a Common Factor of } \frac{(k+1)^2}{12}\right]$$

$$=\frac{(k+1)^2}{12}\left[2k^4+2k^3-k^2+12\left(k^3+3k^2+3k+1\right)\right] \qquad [\text{Expanding Brackets}]$$

$$=\frac{(k+1)^2}{12}\left[2k^4+2k^3-k^2+12k^3+36k^2+36k+12\right]$$

$$=\frac{(k+1)^2}{12}\left[2k^4+14k^3+35k^2+36k+12\right] \qquad [\text{Collecting Like Terms}]$$

Expanding the Right Hand Side of (!) gives:

$$\frac{(k+1)^2(k+2)^2\left(2k^2+6k+3\right)}{12}=\frac{(k+1)^2}{12}\left[(k+2)^2\left(2k^2+6k+3\right)\right]$$

$$=\frac{(k+1)^2}{12}\left[\left(k^2+4k+4\right)\left(2k^2+6k+3\right)\right]$$

$$=\frac{(k+1)^2}{12}\left[2k^4+6k^3+3k^2+8k^3+24k^2+12k+8k^2+24k+12\right]$$

$$\left[\text{Expanding } \left(k^2+4k+4\right)\left(2k^2+6k+3\right)\right]$$

$$=\frac{(k+1)^2}{12}\left[2k^4+14k^3+35k^2+36k+12\right]$$

Hence the Left Hand Side is equal to the Right Hand Side of (!). We have shown $P(k) \quad \Rightarrow \quad P(k+1)$ therefore our given result follows by induction,

$$1^5+2^5+3^5+\ldots+n^5=\frac{n^2(n+1)^2(2n^2+2n-1)}{12}$$

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