Complete Solutions to Exercise 5(c)

1. Since we have $\forall n > N_0$ we have the inequality $\left|\frac{1}{2n}\right| < \varepsilon$ therefore

$$\left|\frac{1}{2n}\right| = \frac{1}{2n} < \frac{1}{2N_0} = \varepsilon$$

Transposing $\frac{1}{2N_0} = \varepsilon$ gives $\frac{1}{2\varepsilon} = N_0$.

For

(a) $\varepsilon = 0.1$ we have $N_0 = \frac{1}{2\varepsilon} = \frac{1}{2\times 0.1} = \frac{1}{0.2} = 5$ and the least natural number $n > N_0$ is $n = 6$.

(b) $\varepsilon = 0.01$ we have $N_0 = \frac{1}{2\varepsilon} = \frac{1}{2\times 0.01} = \frac{1}{0.02} = 50$ and the least natural number $n > N_0$ is $n = 51$.

(c) $\varepsilon = 1\times 10^{-3}$ we have $N_0 = \frac{1}{2\varepsilon} = \frac{1}{2\times\left(1\times 10^{-3}\right)} = \frac{1}{0.002} = 500$ and the least natural number $n > N_0$ is $n = 501$.

(d) $\varepsilon = 1\times 10^{-6}$ we have $N_0 = \frac{1}{2\varepsilon} = \frac{1}{2\times\left(1\times 10^{-6}\right)} = \frac{1}{2\times 10^{-6}} = 500\,000$ and the least natural number $n > N_0$ is $n = 500\,001$.

2. We have $\forall n > N_0$ we have the inequality $\left|\frac{2n+1}{n+1} - 2\right| < \varepsilon$ therefore

$$\begin{aligned}\left|\frac{2n+1}{n+1} - 2\right| &= \left|\frac{2n+1-2(n+1)}{n+1}\right| \\ &= \left|\frac{2n+1-2n-2}{n+1}\right| \\ &= \left|\frac{-1}{n+1}\right| \\ &= \frac{1}{n+1} < \frac{1}{n} < \frac{1}{N_0} = \varepsilon\end{aligned}$$

Transposing gives $N_0 = \frac{1}{\varepsilon}$. For

(a) $\varepsilon = 0.1$ we have $N_0 = \frac{1}{\varepsilon} = \frac{1}{0.1} = 10$ and the least natural number $n > N_0$ is $n = 11$.

(b) $\varepsilon = 0.01$ we have $N_0 = \frac{1}{\varepsilon} = \frac{1}{0.01} = 100$ and the least natural number $n > N_0$ is $n = 101$.

(c) $\varepsilon = 1\times 10^{-3}$ we have $N_0 = \frac{1}{\varepsilon} = \frac{1}{1\times 10^{-3}} = 1000$ and the least natural number $n > N_0$ is $n = 1001$.

(d) $\varepsilon = 1\times10^{-6}$ we have $N_0 = \frac{1}{\varepsilon} = \frac{1}{1\times10^{-6}} = \frac{1}{1\times10^{-6}} = 1\,000\,000$ and the least natural number $n > N_0$ is $n = 1\,000\,001$.

3. (a) We need to prove $\lim_{n\to\infty}\left(\frac{1}{n+1}\right) = 0$

Proof. Let $\varepsilon > 0$ be given. There is a N_0 such that for all $n > N_0$ we have

$$\left|\frac{1}{n+1} - 0\right| = \frac{1}{n+1} < \frac{1}{n} < \frac{1}{N_0} = \varepsilon$$

What is N_0 equal to?

Transposing $\frac{1}{N_0} = \varepsilon$ gives $N_0 = \frac{1}{\varepsilon}$. Since for all $n > N_0 = \frac{1}{\varepsilon}$ we have the inequality $\left|\frac{1}{n+1} - 0\right| < \varepsilon$ therefore this proves $\lim_{n\to\infty}\left(\frac{1}{n+1}\right) = 0$.

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(b) $\lim_{n\to\infty}\left(\frac{1}{n^2+1}\right) = 0$

Proof. Let $\varepsilon > 0$ be given. There is a N_0 such that for all $n > N_0$ we have

$$\left|\frac{1}{n^2+1} - 0\right| = \frac{1}{n^2+1} < \frac{1}{n^2} < \frac{1}{N_0^{\,2}} = \varepsilon$$

What is N_0 equal to?

Transposing $\frac{1}{N_0^{\,2}} = \varepsilon$ gives $N_0 = \frac{1}{\sqrt{\varepsilon}}$. Since for all $n > N_0 = \frac{1}{\sqrt{\varepsilon}}$ we have the inequality $\left|\frac{1}{n^2+1} - 0\right| < \varepsilon$ therefore this proves $\lim_{n\to\infty}\left(\frac{1}{n^2+1}\right) = 0$.

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(c) $\lim_{n\to\infty}\left(\frac{1}{n^3+1}\right) = 0$

Proof. Let $\varepsilon > 0$ be given. There is a N_0 such that for all $n > N_0$ we have

$$\left|\frac{1}{n^3+1} - 0\right| = \frac{1}{n^3+1} < \frac{1}{n^3} < \frac{1}{N_0^{\,3}} = \varepsilon$$

What is N_0 equal to?

Transposing $\frac{1}{N_0^{\,3}} = \varepsilon$ gives $N_0 = \frac{1}{\sqrt[3]{\varepsilon}}$.

Since for all $n > N_0 = \frac{1}{\sqrt[3]{\varepsilon}}$ we have the inequality $\left|\frac{1}{n^3+1} - 0\right| < \varepsilon$ therefore this proves $\lim_{n\to\infty}\left(\frac{1}{n^3+1}\right) = 0$.

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5. (a) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)=1$

Proof. Let $\varepsilon>0$ be given. There is a number N_0 such that for all $n>N_0$ we have

$$\left|1+\frac{1}{n}-1\right|=\left|\frac{1}{n}\right|=\frac{1}{n}<\frac{1}{N_0}=\varepsilon$$

Transposing $\frac{1}{N_0}=\varepsilon$ gives $N_0=\frac{1}{\varepsilon}$. Since for all $n>N_0=\frac{1}{\varepsilon}$ we have the inequality $\left|1+\frac{1}{n}-1\right|<\varepsilon$ therefore this proves $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)=1$.

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(b) $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)=1$

Proof. Let $\varepsilon>0$ be given. There is a number N_0 such that for all $n>N_0$ we have

$$\left|1-\frac{1}{n}-1\right|=\left|-\frac{1}{n}\right|=\frac{1}{n}<\frac{1}{N_0}=\varepsilon$$

Transposing $\frac{1}{N_0}=\varepsilon$ gives $N_0=\frac{1}{\varepsilon}$. Since for all $n>N_0=\frac{1}{\varepsilon}$ we have the inequality $\left|1-\frac{1}{n}-1\right|<\varepsilon$ therefore this proves $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)=1$.

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(c) $\lim_{n\to\infty}\left(9+\frac{1}{n}\right)=9$

Proof. Let $\varepsilon>0$ be given. There is a number N_0 such that for all $n>N_0$ we have

$$\left|9+\frac{1}{n}-9\right|=\left|\frac{1}{n}\right|=\frac{1}{n}<\frac{1}{N_0}=\varepsilon$$

Transposing $\frac{1}{N_0}=\varepsilon$ gives $N_0=\frac{1}{\varepsilon}$. Since for all $n>N_0=\frac{1}{\varepsilon}$ we have the inequality $\left|9+\frac{1}{n}-9\right|<\varepsilon$ therefore this proves $\lim_{n\to\infty}\left(9+\frac{1}{n}\right)=9$.

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(d) $\lim_{n\to\infty}\left(k+\frac{1}{n}\right)=k$ where k is a real number

Proof. Let $\varepsilon>0$ be given. There is a number N_0 such that for all $n>N_0$ we have

$$\left|k+\frac{1}{n}-k\right|=\left|\frac{1}{n}\right|=\frac{1}{n}<\frac{1}{N_0}=\varepsilon$$

Transposing $\frac{1}{N_0}=\varepsilon$ gives $N_0=\frac{1}{\varepsilon}$. Since for all $n>N_0=\frac{1}{\varepsilon}$ we have the inequality

$\left|k+\frac{1}{n}-k\right|<\varepsilon$ therefore this proves $\lim_{n\to\infty}\left(k+\frac{1}{n}\right)=k$.

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6. (a) $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}}\right)=0$

Proof. Let $\varepsilon>0$ be given. There is a N_0 such that for all $n>N_0$ we have

$$\left|\frac{1}{\sqrt{n}}-0\right|=\frac{1}{\sqrt{n}}<\frac{1}{\sqrt{N_0}}=\varepsilon$$

What is N_0 equal to?

Transposing $\frac{1}{\sqrt{N_0}}=\varepsilon$ gives $\sqrt{N_0}=\frac{1}{\varepsilon} \quad\Rightarrow\quad N_0=\frac{1}{\varepsilon^2}$.

Since for all $n>N_0=\frac{1}{\varepsilon^2}$ we have the inequality $\left|\frac{1}{\sqrt{n}}-0\right|<\varepsilon$ therefore this proves $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}}\right)=0$.

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(b) $\lim_{n\to\infty}\left(\frac{1}{\sqrt[k]{n}}\right)=0$ where $k\in\mathbb{N}$

Proof. Let $\varepsilon>0$ be given. There is a N_0 such that for all $n>N_0$ we have

$$\left|\frac{1}{\sqrt[k]{n}}-0\right|=\frac{1}{\sqrt[k]{n}} \qquad (*)$$

Since $n>N_0$ therefore $\sqrt[k]{n}>\sqrt[k]{N_0}$ which implies

$$\frac{1}{\sqrt[k]{n}}<\frac{1}{\sqrt[k]{N_0}} \qquad [\text{Inequality Changes}]$$

Substituting this inequality into (*) we have

$$\left|\frac{1}{\sqrt[k]{n}}-0\right|=\frac{1}{\sqrt[k]{n}}<\frac{1}{\sqrt[k]{N_0}}=\varepsilon$$

What is N_0 equal to?

Transposing $\frac{1}{\sqrt[k]{N_0}}=\varepsilon$ gives $\sqrt[k]{N_0}=\frac{1}{\varepsilon} \quad\Rightarrow\quad N_0=\frac{1}{\varepsilon^k}$.

Since for all $n>N_0=\frac{1}{\varepsilon^k}$ we have the inequality $\left|\frac{1}{\sqrt[k]{n}}-0\right|<\varepsilon$ therefore this proves $\lim_{n\to\infty}\left(\frac{1}{\sqrt[k]{n}}\right)=0$.

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7. (a) $\lim_{n\to\infty}\left(\frac{n^2-1}{n^2+1}\right)=1$

Proof. Let $\varepsilon>0$ be given. There is a N_0 such that for all $n>N_0$ we have

$$\left|\frac{n^2-1}{n^2+1}-1\right|=\left|\frac{n^2-1-\left(n^2+1\right)}{n^2+1}\right|$$
$$=\left|\frac{n^2-1-n^2-1}{n^2+1}\right|$$
$$=\left|\frac{-2}{n^2+1}\right|$$
$$=\frac{|-2|}{n^2+1}=\frac{2}{n^2+1}<\frac{2}{n^2}<\frac{2}{N_0{}^2}=\varepsilon$$

What is N_0 equal to?

Transposing $\frac{2}{N_0{}^2}=\varepsilon$ gives $N_0=\sqrt{\frac{2}{\varepsilon}}$. Since for all $n>N_0=\sqrt{\frac{2}{\varepsilon}}$ we have the inequality $\left|\frac{n^2-1}{n^2+1}-1\right|<\varepsilon$ therefore this proves $\lim_{n\to\infty}\left(\frac{n^2-1}{n^2+1}\right)=1$.

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(b) $\lim_{n\to\infty}\left(\frac{n^2-1}{2^n}\right)=0$

Proof. Let $\varepsilon>0$ be given. There is a number N_0 such that for all $n>N_0$ we have

$$\left|\frac{n^2-1}{2^n}-0\right|=\left|\frac{n^2-1}{2^n}\right|$$
$$=\frac{n^2-1}{2^n}<\frac{n^2}{2^n}$$