Complete Solutions to Exercise 5(d)

1. Prove that if C is a constant then $\lim_{n\to\infty}(C)=C$.

Proof. We use the limit of a sequence definition (5.11). Let $\varepsilon>0$ be given then

$$|C-C|=0<\varepsilon$$

Therefore by (5.11) we have the required result $\lim_{n\to\infty}(C)=C$.

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2. Prove that if K is a constant and $\lim_{n\to\infty}(x_n)$ is a real convergent sequence then $\lim_{n\to\infty}(Kx_n)=K\lim_{n\to\infty}(x_n)$.

Proof. We apply
Proposition (5.15) part (iii) $\lim_{n\to\infty}(x_ny_n)=\lim_{n\to\infty}(x_n)\lim_{n\to\infty}(y_n)$
to $\lim_{n\to\infty}(Kx_n)$. Using this we have

$$\begin{aligned}\lim_{n\to\infty}(Kx_n)&=\lim_{n\to\infty}(K)\lim_{n\to\infty}(x_n)\\&=\underbrace{K}_{\text{By Question 1}}\lim_{n\to\infty}(x_n)\end{aligned}$$

Hence we have our required result.

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3. Let (x_n) and (y_n) be real convergent sequences and α and β be constants. Prove that

$$\lim_{n\to\infty}(\alpha x_n+\beta y_n)=\alpha\lim_{n\to\infty}(x_n)+\beta\lim_{n\to\infty}(y_n)$$

Proof. We apply
Proposition (5.15) part (i) $\lim_{n\to\infty}(x_n+y_n)=\lim_{n\to\infty}(x_n)+\lim_{n\to\infty}(y_n)$
We have

$$\begin{aligned}\lim_{n\to\infty}(\alpha x_n+\beta y_n)&=\lim_{n\to\infty}(\alpha x_n)+\lim_{n\to\infty}(\beta y_n)\\&=\alpha\lim_{n\to\infty}(x_n)+\beta\lim_{n\to\infty}(y_n)\qquad[\text{By Question 2}]\end{aligned}$$

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4. Let (x_n), (y_n) and (z_n) be real convergent sequences such that

$$\lim_{n\to\infty}(x_n)=L,\ \lim_{n\to\infty}(y_n)=M\ \text{ and }\ \lim_{n\to\infty}(z_n)=K$$

Prove that $\lim_{n\to\infty}(x_n+y_n+z_n)=L+M+K$.

Proof. Required to prove $|(x_n+y_n+z_n)-(L+M+N)|<\varepsilon$ for an arbitrary $\varepsilon>0$ and large enough n.

(5.11) $\lim_{n\to\infty}(x_n)=L \iff \forall\varepsilon>0\ \ \exists N_0\in\mathbb{N}$ such that $\forall n>N_0\ \ |x_n-L|<\varepsilon$

Let $\varepsilon > 0$ be arbitrary. Since we are given $\lim_{n\to\infty}(x_n) = L$, $\lim_{n\to\infty}(y_n) = M$ and $\lim_{n\to\infty}(z_n) = K$ therefore there are numbers N_0, N_1 and N_2 such that for all $n > N_0$, $n > N_1$ and $n > N_2$ respectively we have

$$|x_n - L| < \varepsilon_1,\ |y_n - M| < \varepsilon_2 \text{ and } |z_n - K| < \varepsilon_3 \qquad (*)$$

where ε_1, ε_2 and ε_3 are positive real numbers to be determined later on. Let $N = \max\{N_0, N_1, N_2\}$ which means that N is the largest value out of N_0, N_1 and N_2. Consider the term $|(x_n + y_n + z_n) - (L + M + N)|$ then for all $n > N$ we have

$$\begin{aligned} |(x_n + y_n + z_n) - (L + M + N)| &= |(x_n - L) + (y_n - M) + (z_n - K)| \\ &\le |x_n - L| + |y_n - M| + |z_n - K| \quad \begin{bmatrix}\text{By Triangle Inequality} \\ |a+b+c| \le |a| + |b| + |c|\end{bmatrix} \\ &< \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \quad [\text{By } (*)] \end{aligned}$$

We have $|(x_n + y_n + z_n) - (L + M + N)| < \varepsilon_1 + \varepsilon_2 + \varepsilon_3$ but we need to show $|(x_n + y_n + z_n) - (L + M + N)| < \varepsilon$. *How?*

Let $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \frac{\varepsilon}{3}$. Therefore from above we have

$$\begin{aligned} |(x_n + y_n + z_n) - (L + M + N)| &< \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \\ &= \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

Hence we have the required result that is $|(x_n + y_n + z_n) - (L + M + N)| < \varepsilon$ for $n > N$ and arbitrary $\varepsilon > 0$ which means that $\lim_{n\to\infty}(x_n + y_n + z_n) = L + M + K$.

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5. Let (x_n) be a real convergent sequence such that $\lim_{n\to\infty}(x_n) = L$. Prove that $\lim_{n\to\infty}(-x_n) = -L$

Proof. Since we are given $\lim_{n\to\infty}(x_n) = L$ therefore for any $\varepsilon > 0$ there is a number N_0 such that for all $n > N_0$ we have

$$|x_n - L| < \varepsilon$$

We need to show $|-x_n - (-L)| < \varepsilon$. *How?*

Consider $|-x_n - (-L)|$:

$$\begin{aligned} |-x_n - (-L)| &= |-(x_n - L)| \\ &= |-1(x_n - L)| \\ &= \underbrace{|-1|}_{=1}|x_n - L| \qquad [\text{Because } |ab| = |a||b|] \\ &= |x_n - L| < \varepsilon \end{aligned}$$

Since $\left|-x_n-(-L)\right|<\varepsilon$ for all $n>N_0$ and arbitrary $\varepsilon>0$ therefore by (5.11) we have $\lim_{n\to\infty}(-x_n)=-L$.

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6. Prove Proposition (5.15) part (ii):

Let (x_n) and (y_n) be real convergent sequences such that $\lim_{n\to\infty}(x_n)=L$ and $\lim_{n\to\infty}(y_n)=M$. Then

$$\lim_{n\to\infty}(x_n-y_n)=L-M$$

Proof. We can use the result of question 3 by rewriting x_n-y_n as $x_n+(-1)y_n$:

$$\begin{aligned}\lim_{n\to\infty}(x_n-y_n)&=\lim_{n\to\infty}(x_n+(-1)y_n)\\&=\lim_{n\to\infty}(x_n)+(-1)\lim_{n\to\infty}(y_n)\qquad\text{[By Result of Question 3]}\\&=\lim_{n\to\infty}(x_n)-\lim_{n\to\infty}(y_n)\end{aligned}$$

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7. **WORKBOOK QUESTION.**

8. Give an example of a sequence which is bounded but **not** convergent.

Solution. Let $x_n=(-1)^n$ then $|x_n|\le 1$ but the sequence (x_n) is **not convergent.**

9. Let (x_n) be a real convergent sequence such that $\lim_{n\to\infty}(x_n)=L$. Prove that

$$\lim_{n\to\infty}(|x_n|)=|L|$$

Proof. Let $\varepsilon>0$ be given. *What do we need to prove?*

$\left||x_n|-|L|\right|<\varepsilon$ for $n>N_0$. *How?*

Since we are given $\lim_{n\to\infty}(x_n)=L$ therefore there is a number N_0 such that for all $n>N_0$ we have the inequality

$$|x_n-L|<\varepsilon \qquad (*)$$

Consider the term $\left||x_n|-|L|\right|$, then for $n>N_0$ we have

$$\begin{aligned}\left||x_n|-|L|\right|&\le|x_n-L|\qquad\left[\text{Using the Inequality }\left||a|-|b|\right|\le|a-b|\right]\\&<\varepsilon\qquad\left[\text{By }(*)\right]\end{aligned}$$

Hence we have our required result $\left||x_n|-|L|\right|<\varepsilon$ therefore $\lim_{n\to\infty}(|x_n|)=|L|$.

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(5.11) $\lim_{n\to\infty}(x_n)=L \iff \forall\varepsilon>0\ \exists N_0\in\mathbb{N}$ such that $\forall n>N_0\quad |x_n-L|<\varepsilon$

Result of Question 3 $\quad\lim_{n\to\infty}(\alpha x_n+\beta y_n)=\alpha\lim_{n\to\infty}(x_n)+\beta\lim_{n\to\infty}(y_n)$

10. Let both (x_n) and (y_n) be real convergent sequences. Prove that if for all $n \in \mathbb{N}$ we have the inequality $x_n \le y_n$ then $\lim_{n\to\infty}(x_n) \le \lim_{n\to\infty}(y_n)$.

Proof. Very similar to proof of proposition (5.17).

11. Let (x_n) be a real convergent sequence such that $\lim_{n\to\infty}(x_n) = L$. If for all $n \in \mathbb{N}$ we have $K \le x_n \le M$ where K and M are real numbers then prove that $K \le L \le M$.

Proof. By the result of question 10 we have $\lim_{n\to\infty}(x_n) \le \lim_{n\to\infty}(M) = M$ because M is the constant sequence. Again by the result of question 10 we have the other inequality, that is $\lim_{n\to\infty}(K) \le \lim_{n\to\infty}(x_n)$. Since K is a constant sequence therefore $K = \lim_{n\to\infty}(K) \le \lim_{n\to\infty}(x_n)$. Combining these two inequalities together we have our result, $K \le \lim_{n\to\infty}(x_n) = L \le M$.

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12. WORKBOOK QUESTION.(**Sandwich Rule**). Let (x_n), (y_n) and (z_n) be real convergent sequences such that for all $n \in \mathbb{N}$ we have $x_n \le y_n \le z_n$. If $\lim_{n\to\infty}(x_n) = \lim_{n\to\infty}(z_n) = L$ prove that $\lim_{n\to\infty}(y_n) = L$. (This is also called the **Squeeze Theorem**).

13. Let (x_n) be a real convergent sequence such that $\lim_{n\to\infty}(x_n) = 0$ and (y_n) be a bounded sequence. Prove that $\lim_{n\to\infty}(x_n y_n) = 0$.

Proof. Since we are given $\lim_{n\to\infty}(x_n) = 0$ therefore there is a $N_0 \in \mathbb{N}$ such that for all $n > N_0$ we have

$$|x_n - 0| = |x_n| < \varepsilon_1$$

Since (y_n) is a bounded sequence therefore there is a real number $K > 0$ such that for all $n \in \mathbb{N}$ the following inequality holds, $|y_n| \le K$.

What do we need to prove?

Just need to show that $|x_n y_n - 0| < \varepsilon$ where $\varepsilon > 0$ is arbitrary. Consider the term $|x_n y_n - 0|$, then for all $n > N_0$ we have

$$|x_n y_n - 0| = |x_n||y_n| \le |x_n| K < \varepsilon_1 K$$

Hence we have $|x_n y_n - 0| < \varepsilon_1 K$ *but how do we show* $|x_n y_n - 0| < \varepsilon$ *?*

By letting $\varepsilon_1 = \dfrac{\varepsilon}{K}$ then

$$|x_n y_n - 0| < \varepsilon_1 K = \frac{\varepsilon}{K} K = \varepsilon \quad \text{[Cancelling } K\text{'s]}$$

Hence $|x_n y_n - 0| < \varepsilon$ which gives the required result, $\lim_{n\to\infty}(x_n y_n) = 0$. ■

Question 10 If $x_n \le y_n$ then $\lim_{n\to\infty}(x_n) \le \lim_{n\to\infty}(y_n)$

14. Let (x_n) be a real convergent sequence such that $\lim_{n\to\infty}(x_n) = L$. Prove that

$$\lim_{n\to\infty}\left(x_n^{\,2}\right) = L^2$$

By
(i) using proposition (5.15)
(ii) using the formal definition of the limit (5.11).

Proof. (i) Applying proposition (5.15) part (iii)

$$\lim_{n\to\infty}(x_n y_n) = \lim_{n\to\infty}(x_n)\lim_{n\to\infty}(y_n)$$

to $\lim_{n\to\infty}\left(x_n^{\,2}\right)$ we have

$$\begin{aligned}\lim_{n\to\infty}\left(x_n^{\,2}\right) = \lim_{n\to\infty}(x_n x_n) &= \lim_{n\to\infty}(x_n)\lim_{n\to\infty}(x_n) && \left[\text{By (5.15) part (iii)}\right]\\ &= L^2 && \left[\text{Because } \lim_{n\to\infty}(x_n) = L^2\right]\end{aligned}$$

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Proof. (ii) Let $\varepsilon > 0$ be given. Since we have $\lim_{n\to\infty}(x_n) = L$ therefore there is a number N_0 such that for all $n > N_0$ we have the inequality

$$|x_n - L| < \varepsilon_1 \qquad (*)$$

where ε_1 is a positive real number and is determined later on. *What do we need to prove?*

Required to prove $\left|x_n^{\,2} - L^2\right| < \varepsilon$. Consider the term $\left|x_n^{\,2} - L^2\right|$, then for all $n > N_0$ we have

$$\begin{aligned}\left|x_n^{\,2} - L^2\right| &= |x_n - L||x_n + L| && \left[\text{Using } a^2 - b^2 = (a-b)(a+b)\right]\\ &< \varepsilon_1|x_n + L| && \left[\text{By } (*)\ |x_n - L| < \varepsilon_1\right]\\ &\le \varepsilon_1\left(|x_n| + |L|\right) && \left[\text{By the Triangle Inequality } |a+b| \le |a| + |b|\right]\end{aligned}$$

So far we have

$$\left|x_n^{\,2} - L^2\right| < \varepsilon_1\left(|x_n| + |L|\right) \qquad (\dagger)$$

How do we get $\left|x_n^{\,2} - L^2\right| < \varepsilon$ *?*

Since the sequence (x_n) is convergent therefore it is bounded which means there is a real number $K > 0$ such that for all $n \in \mathbb{N}$

$$|x_n| \le K$$

Putting this, $|x_n| < K$, into (†) gives

$$\left|x_n^{\,2} - L^2\right| < \varepsilon_1\left(|x_n| + |L|\right) \le \varepsilon_1\left(K + |L|\right)$$

For $\left|x_n^{\,2} - L^2\right| < \varepsilon$ *what do we take* ε_1 *to equal?*

$\varepsilon_1 = \dfrac{\varepsilon}{K + |L|}$. Hence we have

$$\left|x_n^{\,2} - L^2\right| < \varepsilon_1\left(K + |L|\right) = \frac{\varepsilon}{\left(K + |L|\right)}\left(K + |L|\right) = \varepsilon$$

which is our required result. ■

15. Let (x_n) and be a real convergent sequence such that $\lim_{n\to\infty}(x_n) = L$. Prove that

$$\lim_{n\to\infty}\left(x_n{}^m\right) = L^m \quad \text{where } m \in \mathbb{N}$$

Proof. We use induction because we want to prove the result for the natural numbers $\mathbb{N}$.
We first check the result for $m = 1$:

$$\lim_{n\to\infty}\left(x_n{}^1\right) = \lim_{n\to\infty}(x_n) = L = L^1 \qquad \surd$$

Assume the result is true for $m = k$ that is

$$\lim_{n\to\infty}\left(x_n{}^k\right) = L^k \qquad (\$)$$

Consider the case $m = k+1$. *What do we need to prove?*
Required to prove $\lim_{n\to\infty}\left(x_n{}^{k+1}\right) = L^{k+1}$. *How?*
Using proposition (5.15) part (iii):

$$\lim_{n\to\infty}(x_n y_n) = \lim_{n\to\infty}(x_n)\lim_{n\to\infty}(y_n)$$

We have

$$\begin{aligned}\lim_{n\to\infty}\left(x_n{}^{k+1}\right) &= \lim_{n\to\infty}\left(x_n{}^k x_n\right) \\ &= \lim_{n\to\infty}\left(x_n{}^k\right)\lim_{n\to\infty}(x_n) && \left[\text{By (5.15) part(iii)}\right] \\ &= L^k L && \left[\text{By (\$)}\right] \\ &= L^{k+1}\end{aligned}$$

Hence we have shown $\lim_{n\to\infty}\left(x_n{}^{k+1}\right) = L^{k+1}$ therefore the given result follows by induction.

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16. Let (x_n) be a real convergent sequence such that $\lim_{n\to\infty}(x_n) = L$ and for all $n \in \mathbb{N}$ $x_n \geq 0$. Prove that

$$\lim_{n\to\infty}\left(\sqrt{x_n}\right) = \sqrt{L}$$

[Hint: $\sqrt{a} - \sqrt{b} = \dfrac{a-b}{\sqrt{a}+\sqrt{b}}$]

Proof. Since for all $n \in \mathbb{N}$ $x_n \geq 0$ therefore $\lim_{n\to\infty}(x_n) = L \geq 0$. We consider two cases when $L = 0$ and $L > 0$.
Case (I) Let $L = 0$ and $\varepsilon > 0$ be given. There exists a $N_0 \in \mathbb{N}$ such that for all $n > N_0$ we have

$$|x_n - L| < \varepsilon_1$$

Since $L = 0$ we have

$$|x_n - 0| = |x_n| = x_n < \varepsilon_1 \qquad (\dagger)$$

What do we need to prove?
Required to prove that $\sqrt{x_n} < \varepsilon$. *How?*
By (†) we have $x_n < \varepsilon_1$. Let $\varepsilon_1 = \varepsilon^2$ then for all for all $n > N_0$ we have

$$0 \le \left|\sqrt{x_n} - 0\right| = \sqrt{x_n} < \sqrt{\varepsilon_1} = \sqrt{\varepsilon^2} = \varepsilon$$

Hence for all $n > N_0$ we have $0 \le \sqrt{x_n} < \varepsilon$ therefore $\lim_{n\to\infty}\left(\sqrt{x_n}\right) = 0 = \sqrt{L}$.

Case (II) $L > 0$. Let $\varepsilon > 0$ be given. Then there exists a $N_1 \in \mathbb{N}$ such that for all $n > N_1$ we have

$$|x_n - L| < \varepsilon_2 \qquad (\dagger\dagger)$$

What do we need to prove?

Required to prove $\left|\sqrt{x_n} - \sqrt{L}\right| < \varepsilon$ for $\varepsilon > 0$ and $n > N_1$ which gives $\lim_{n\to\infty}\left(\sqrt{x_n}\right) = \sqrt{L}$. Consider the term $\left|\sqrt{x_n} - \sqrt{L}\right|$, then for all $n > N_1$

$$\left|\sqrt{x_n} - \sqrt{L}\right| = \frac{|x_n - L|}{\left|\sqrt{x_n} + \sqrt{L}\right|} \qquad \left[\text{Using } \sqrt{a} - \sqrt{b} = \frac{a-b}{\sqrt{a}+\sqrt{b}}\right]$$

$$< \frac{\varepsilon_2}{\left|\sqrt{x_n} + \sqrt{L}\right|} \qquad \left[\text{By } (\dagger\dagger) \text{ on Numerator}\right]$$

$$\le \frac{\varepsilon_2}{\left|\sqrt{L}\right|} \qquad \left[\text{Because } \sqrt{x_n} + \sqrt{L} \ge \sqrt{L}\right]$$

We have $\left|\sqrt{x_n} - \sqrt{L}\right| < \frac{\varepsilon_2}{\sqrt{L}}$ but we need $\left|\sqrt{x_n} - \sqrt{L}\right| < \varepsilon$. *How do we achieve this result?*

By letting $\varepsilon_2 = \sqrt{L}\varepsilon$. For all $n > N_1$ we have

$$\left|\sqrt{x_n} - \sqrt{L}\right| < \frac{\varepsilon_2}{\sqrt{L}} = \frac{\sqrt{L}\varepsilon}{\sqrt{L}} = \varepsilon$$

Hence we have our required result $\left|\sqrt{x_n} - \sqrt{L}\right| < \varepsilon$ which gives $\lim_{n\to\infty}\left(\sqrt{x_n}\right) = \sqrt{L}$.

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17. Let (x_n) be a real convergent sequence such that $\lim_{n\to\infty}(x_n) = L$. Prove that

$$\lim_{n\to\infty}\left(\cos(x_n)\right) = \cos(L)$$

[Hint: For all $x \in \mathbb{R}$ we have the inequality $|\sin(x)| \le |x|$].

Proof. Let $\varepsilon > 0$ be arbitrary. Since we are given $\lim_{n\to\infty}(x_n) = L$ therefore there is a $N_0 \in \mathbb{N}$ such that for all $n > N_0$ we have

$$|x_n - L| < \varepsilon \qquad (*)$$

What do we need to prove?

Required to prove $|\cos(x_n) - \cos(L)| < \varepsilon$. Consider the term $|\cos(x_n) - \cos(L)|$, then for all $n > N_0$ we have

$$\left|\cos(x_n)-\cos(L)\right| = \left|-2\sin\left(\frac{x_n+L}{2}\right)\sin\left(\frac{x_n-L}{2}\right)\right| \quad \left[\begin{array}{l}\text{By Using the}\\ \text{Trigonometric Identity}\\ \cos(A)-\cos(B)\\ \quad =-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\end{array}\right]$$

$$= \left|-2\right|\left|\sin\left(\frac{x_n+L}{2}\right)\right|\left|\sin\left(\frac{x_n-L}{2}\right)\right| \quad \left[\text{Applying } |abc|=|a||b||c|\right]$$

$$\le 2|1|\left|\frac{x_n-L}{2}\right| \quad \left[\begin{array}{l}\text{Because } |-2|=2,\\ \left|\sin\left(\frac{x_n+L}{2}\right)\right|\le 1 \text{ and using}\\ \text{the hint on the last term}\end{array}\right]$$

$$= \left|x_n-L\right| \quad \left[\text{Cancelling 2's}\right]$$

Hence $\left|\cos(x_n)-\cos(L)\right| \le \left|x_n-L\right| < \underbrace{\varepsilon}_{\text{By (*)}}$. Therefore we have our required result $\left|\cos(x_n)-\cos(L)\right| < \varepsilon$ for $n > N_0$ which means that $\lim_{n\to\infty}\left(\cos(x_n)\right) = \cos(L)$.

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