Exercise 2c

1. Show that the following series diverge:

   (a) $\sum_{n=1}^{\infty}\left(\frac{2n-1}{n}\right)$ (b) $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)$ (c) $\sum_{n=1}^{\infty}\left(\frac{2n^2-n+1}{n^2+n+1}\right)$

   (d) $\sum_{n=1}^{\infty}\left(\frac{3n+1}{5n-1}\right)$ (e) $\sum_{n=1}^{\infty}(2)^n$ (f) $\sum_{n=1}^{\infty}\left(\frac{7}{6}\right)^n$

   (g) $\sum_{n=1}^{\infty}\left(\frac{\sqrt{n}-1}{\sqrt{n}+1}\right)$ (h) $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)$ (i) $\sum_{n=1}^{\infty}\cos(n\pi)$

   (j) $\sum_{n=1}^{\infty}\sin(n\pi)$

2. Show that the series $\sum_{n=1}^{\infty}\left(\sqrt{\frac{n-1}{n+1}}\right)$ diverges.

3. Prove that if both $\sum(a_k)$ and $\sum(b_k)$ are convergent then

   (a) $\sum(a_k-b_k)=\sum(a_k)-\sum(b_k)$

   (b) $\sum c(a_k)=c\sum(a_k)$ where c is a constant

   (c) $\sum(ca_k+db_k)=c\sum(a_k)+d\sum(b_k)$ where c and d are constants.

4. Find the **first error** in the following derivation:
   Let both $\sum(a_k)$ and $\sum(b_k)$ be convergent then

$$\begin{aligned}\sum_{k=1}^{\infty}(a_kb_k) &= \lim_{n\to\infty}\left[\sum_{k=1}^{n}(a_kb_k)\right]\\ &= \lim_{n\to\infty}\left[\sum_{k=1}^{n}(a_k)\sum_{k=1}^{n}(b_k)\right]\\ &= \lim_{n\to\infty}\left[\sum_{k=1}^{n}(a_k)\right]\lim_{n\to\infty}\left[\sum_{k=1}^{n}(b_k)\right]\\ &= \sum_{k=1}^{\infty}(a_k)\sum_{k=1}^{\infty}(b_k)\end{aligned}$$

5. Test the following series for convergence. If the series converges then determine its sum.

   (a) $\sum_{n=1}^{\infty}\left(e^{-n}+\pi^{-n}\right)$

   (b) $\sum_{n=1}^{\infty}\left[\frac{1}{4n^2-1}+\left(\frac{2}{3}\right)^n\right]$

   (c) $\sum_{n=1}^{\infty}\left(\frac{4^{n-2}+6^{n-1}}{12^n}\right)$

(d) $\sum_{n=1}^{\infty}\left(\frac{1}{n^2+n}+e^{-2n}\right)$

(e) $\sum_{n=1}^{\infty}\left(\frac{4}{3n(n+2)}+2\pi^{-2n}\right)$

6. Show that the following series diverge:

   (a) $\sum_{n=1}^{\infty}\left(\cos^2(x)+\sin^2(x)\right)^n$ where $x\in\mathbb{R}$

   (b) $\sum_{n=1}^{\infty}\left(e^{i\pi}\right)^n$

7. Prove that if $|x|<1$ then $\lim_{n\to\infty}\left(x^n\right)=0$.

8. Prove that $\sum_{n=1}^{\infty}\left(n^{1/n}\right)$ diverges.

9. Show that $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n$ diverges.

Solutions 2c

1. All the series diverge because for each case $\lim_{n\to\infty}(a_n)$ is equal to the following values and none of them are zero:

   (a) 2 (b) 1 (c) 2 (d) 3/5

   For (e) and (f) the limits do not exist.

   (g) 1 (h) 1

   For (i) and (j) the limits do not exist.

2. Since $\lim_{n\to\infty}\left(\sqrt{\frac{n-1}{n+1}}\right)=1$ therefore the series diverges.
3. Very similar to the proofs under section C.
4. Error in the second line because

$$\sum_{k=1}^{n}(a_k b_k)\neq\sum_{k=1}^{n}(a_k)\sum_{k=1}^{n}(b_k)$$

   Examine the notation carefully and see what it means on both sides of the $\neq$ sign.

5. All converge with the following values:

   (a) $\frac{\pi+e-2}{(e-1)(\pi-1)}$ (b) 5/2 (c) 19/96 (d) $\frac{e^2}{e^2-1}$ (e) $\frac{\pi^2+1}{\pi^2-1}$

6. (a) $\cos^2(x)+\sin^2(x)=1$ therefore we have $\sum_{n=1}^{\infty}(1)^n$. Hence $\lim_{n\to\infty}(1)^n=1\neq 0$.

   (b) $\sum_{n=1}^{\infty}\left(e^{i\pi}\right)^n=\sum_{n=1}^{\infty}(-1)^n$ and $\lim_{n\to\infty}(-1)^n$ does not exist.

7. Consider the geometric series

$$\sum_{n=1}^{\infty}\left(x^{n}\right)=x+x^{2}+x^{3}+\ldots$$

Common ratio $|r|=|x|<1$.

8. Let $n=1+k_n$ and we can express the given nth term as

$$(n)^{1/n}=(1+k_n)^{1/n}$$

Expand the Right Hand Side by using the binomial and show $\lim_{n\to\infty}\left(n^{1/n}\right)=1$.

9. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e\neq 0$.