Exercise 2d

Throughout this exercise $\sum$ represents $\sum_{n=1}^{\infty}$.

1. Discuss convergence or divergence for each of the following series:

(a) $\sum\left(\frac{1}{n^3}\right)$ (b) $\sum\left(\frac{1}{n^4}\right)$ (c) $\sum\left(\frac{1}{n^{1/3}}\right)$ (d) $\sum\left(\frac{1}{n^e}\right)$

(e) $\sum\left(\frac{1}{n^\pi}\right)$ (f) $\sum\left(\frac{1}{n^{\cos^2(x)+\sin^2(x)}}\right)$ where $x \in \mathbb{R}$

(g) $\sum\left(\frac{1}{\sqrt{n}\sqrt[3]{n}}\right)$ (h) $\sum\left(\frac{\sqrt[3]{n}}{\sqrt{n}}\right)$ (i) $\sum\left(\frac{\sqrt{n}}{n^2}\right)$

(j) $\sum\left(\frac{1}{n^{\sqrt{n}}}\right)$

2. Test each of the following series for convergence:

(a) $\sum\left(\frac{1}{n^2+n}\right)$ (b) $\sum\left(\frac{1}{3^n+n}\right)$ (c) $\sum\left(\frac{1+4^n}{3^n}\right)$

(d) $\sum\left(\frac{1}{1+4^n}\right)$ (e) $\sum\left(\frac{1}{3^n-1}\right)$ (f) $\sum\left(\frac{1}{n!n}\right)$

(g) $\sum\left(\frac{1}{\sqrt{n-1}}\right)$ (h) $\sum\left(\frac{1}{5^n n!}\right)$ (i) $\sum\left(\frac{1}{2^n+2n}\right)$

(j) $\sum\left(\frac{n^3+1}{n^5+1}\right)$ (k) $\sum\left(\frac{n+2^n}{3^n}\right)$

3. Discuss the convergence or divergence of each of the following series:

(a) $\sum_{n=2}^{\infty}\left(\frac{1}{n^3-n}\right)$ (b) $\sum\left(\frac{1}{4^n-1}\right)$ (c) $\sum\left(\frac{1}{\sqrt{n+2}}\right)$

(d) $\sum\left(\frac{n}{n^2-3n+5}\right)$ (e) $\sum\left(\frac{n+1}{n^2+n}\right)$

(f) $\sum\left(\frac{2n^2+n-2}{n^3+3n}\right)$ (g) $\sum\left(\frac{n}{5n^2-1}\right)$

4. The following is a corollary of the Normal Comparison test (2.12).
Let (a_n) and (b_n) be real sequences. Prove that if for some $M \in \mathbb{N}$

$$0 \le a_n \le b_n \qquad \forall n \ge M$$

then $\sum(b_n)$ is convergent implies $\sum(a_n)$ is convergent.
(**Remark**. This is same as the comparison test but the b_n terms are greater than or equal to the a_n terms starting from $n = M$ rather than $n = 1$. It is a **weaker form** of the comparison test because the above inequality does **not** need to be valid for all the natural numbers n).

5. (a) Show that $\sum\left(n^{-n}\right)$ converges.

   (b) Show that $\sum_{n=2}^{\infty}\left(\frac{1}{n^2\ln(n)}\right)$ converges.

6. Discuss the convergence or divergence of the following series by applying the normal comparison test:

   (a) $\sum_{n=2}^{\infty}\left(\frac{1}{\ln(n)}\right)$ (b) $\sum\left(\frac{1}{\sin(n)}\right)$ (c) $\sum\left(\frac{\cos(n)}{n^2}\right)$

   (d) $\sum\left(\frac{\sin(n)}{n\sqrt{n}}\right)$ (e) $\sum\left(\frac{\ln(n)}{n}\right)$ (f) $\sum\left(\frac{1}{e^n}\right)$

   (g) $\sum_{n=3}^{\infty}\left(\frac{1}{\sqrt{n^2-3n+1}}\right)$ (h) $\sum\left(\frac{1}{\sqrt{1+n^2}-n}\right)$

   (i) $\sum\left(\sqrt{1+n^2}-n\right)$

7. Let a_n be real. Prove that if $\sum|a_n|$ converges then $\sum(a_n)$ converges. (**Remark**. If $\sum|a_n|$ is convergent then the series $\sum(a_n)$ is said to be absolutely convergent).

8. Show that if $\lim_{n\to\infty}(nu_n)=L\neq 0$ then $\sum(u_n)$ diverges where $u_n>0$ is real.

9. Prove the following result:
   Let (a_n) and (b_n) be real sequences such that for some constant $M\in\mathbb{R}$ we have

   $$0\leq a_n\leq Mb_n \qquad \forall n\geq K \quad \text{where } K\in\mathbb{N}$$

   Then $\sum(b_n)$ is convergent implies $\sum(a_n)$ is convergent.
   (**Remark**: This is another weaker version of the comparison test. Our normal comparison test is the case with $M=K=1$).

10. Let (a_n) and (b_n) be real positive sequences. Show that

    (i) If $\sum(a_n+b_n)$ is convergent then $\sum\left(\sqrt{a_nb_n}\right)$ is convergent.

    (ii) If $\sum(a_n)$ is convergent then for $m\in\mathbb{N}$, $\sum\left(\sqrt{a_na_{n+m}}\right)$ is also convergent.

11. Let (a_n) be real positive sequence. Prove the following results:

    (a) If $\sum(a_n)$ converges then $\sum(\ln(a_n))$ converges

    (b) If $\sum(a_n)$ diverges then $\sum\left(e^{a_n}\right)$ diverges.

12. Let (a_n) be real positive sequence. Prove that if $\sum(a_n)$ is convergent then $\sum(a_n)^2$ is also convergent.

13. Give an example of a convergent series $\sum(a_n)$ but $\sum\left(\sqrt{a_n}\right)$ diverges where $a_n > 0$.

14. Prove the Limit Comparison test (2.13).

Solutions

1. Use the p-series test in each case.
(a) Converges because p=3
(b) Converges because p=4
(c) Diverges because p=1/3
(d) Converges because $p = e > 1$
(e) Converges because $p = \pi > 1$
(f) Diverges because $p = \cos^2(x) + \sin^2(x) = 1$
(g) Diverges. By using the rules of indices we have $p = 1/6 \le 1$.
(h) Diverges. By using the rules of indices we have $p = 5/6 \le 1$.
(i) Converges. By using the rules of indices we have $p = 3/2 > 1$.
(j) Converges because for $n \ge 2$ we have $\sqrt{n} \ge \sqrt{2} > 1$

2. Use the normal comparison test for each example. The following comparisons are only suggestions.
(a) Converges. Compare with $1/n^2$.
(b) Converges. Compare with $1/3^n$.
(c) Diverges. Compare with $(4/3)^n$.
(d) Converges. Compare with $1/4^n$.
(e) Diverges. Compare with $1/3^n$.
(f) Converges. Compare with $1/n^2$.
(g) Diverges. Compare with $1/\sqrt{n}$.
(h) Converges. Compare with $1/5^n$.
(i) Converges. Compare with $1/2^n$.
(j) Converges. Rewrite the series first and then compare each part with $1/n^2$.
(k) Converges. Compare with $(2/3)^n$.

3. Use the Limit Comparison test in all cases apart from (e). The following comparisons are only suggestions.
(a) Converges. Compare with $1/n^3$.
(b) Converges. Compare with $1/(4^n - 1)$.
(c) Diverges. Compare with $1/\sqrt{n}$.
(d) Diverges. Compare with $1/n$.
(e) Diverges. Simplify the given expression.
(f) Diverges. Compare with $1/n$.

(g) Diverges. Compare with $1/n$.

4. Very similar to the proof of (2.12). To prove that $\sum(a_n)$ is convergent separate them into $a_1 + a_2 + a_3 + ... + a_{M-1} + \sum_{k=M}^{\infty}(a_k)$.

5. We use comparison test of question 4.

(a) Compare with $1/n^2$.

(b) Compare with $1/n^2$.

6. Use the normal comparison test (or it's corollary in question 4) for each example. The following comparisons are only suggestions.

(a) Diverges. Compare with $1/n$.

(b) Diverges. Compare with $1/n$.

(c) Converges. Compare with $1/n^2$ because $\cos(n) \le 1$.

(d) Converges. Compare with $1/n^{3/2}$.

(e) Diverges. Compare with $1/n$.

(f) Converges. Compare with $2/n^2$ because $e^n = 1 + n + \frac{n^2}{2!} + \frac{n^3}{3!} + ... > \frac{n^2}{2}$

(g) Diverges. Compare with $1/(n-1)$ by using completing the square method.

(h) Diverges. Compare with 1 by completing the square.

(i) Diverges. Compare with -1 by completing the square.

7. Use $a_n \le |a_n|$ and then apply the comparison test.

8. Apply the limit comparison test with $a_n = u_n$ and $b_n = 1/n$.

9. Assume $\sum(b_n)$ converges and then apply the normal comparison test (2.12) on the given inequality.

10. (i) Use the Arithmetic and Geometric mean inequality:

$$\sqrt{a_n b_n} \le \frac{a_n + b_n}{2}$$

and then apply the comparison test.

(ii) Similar to (i)

11. (a) We use the following inequality:

$$\ln(x) \le x \quad \text{where } x \in \mathbb{R}^+$$

and then apply the comparison test.

(b) We use the following inequality:

$$e^x > x \quad \text{where } x \in \mathbb{R}$$

and then apply the comparison test.

12. $\sum(a_n)$ converges then for some $K \in \mathbb{N}$ we have $\forall n \ge K \;\; {a_n}^2 < a_n$. Now apply the comparison test.

13. Consider $a_n = 1/n^2$ and then $a_n = 1/n$.

14. By using an inequality from limits of sequences we have $\frac{1}{2}L < \frac{a_n}{b_n} < 2L$ and then apply the comparison test.