## Exercise 2g

1. Test the following series for convergence:

(a) $\sum\left(\frac{(-1)^{n+1}}{n}\right)$ [Alternating Harmonic Series]

(b) $\sum\left(\frac{(-1)^{n+1}}{5n-1}\right)$ (c) $\sum\left(\frac{(-1)^{n+1}}{6n+2}\right)$ (d) $\sum\left(\frac{(-1)^{n+1}}{2^n}\right)$

(e) $\sum\left(\frac{(-1)^{n+1}n}{n+1}\right)$ (f) $\sum\left(\frac{(-1)^{n+1}}{2n-1}\right)$ (g) $\sum\left((-1)^{n+1}n!\right)$

(h) $\sum\left(\frac{(-1)^{n+1}n^2}{n^2+1}\right)$ (i) $\sum\left(\frac{(-1)^{n+1}}{\sqrt{n}}\right)$ (j) $\sum\left(\frac{(-1)^{n+1}n}{2^n}\right)$

(k) $\sum_{n=2}^{\infty}\left(\frac{(-1)^{n+1}}{n\ln(n)}\right)$

2. (a) Show that the exponential function, $e^x$, is an increasing function on $\mathbb{R}$.
(b) Show that the decaying exponential function, $e^{-x}$, is a decreasing function on $\mathbb{R}$.
(c) Show that $\sin(x)$ is a decreasing function in the interval $\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$.
(d) Show that $\cos(x)$ is an increasing function in the interval $[\pi,2\pi]$.
(e) Show that $\frac{1}{x}$ is a decreasing function for $x \neq 0$.
(f) Show that $\frac{x}{x^2+1}$ is a decreasing function for $x \geq 1$.
(g) Show that $\cosh(x)$ is an increasing function for $x \geq 0$.
(h) Show that $xe^x$ is an increasing function for $x \geq 1$.

3. Show that the following series converge:

(a) $\sum\left((-1)^{n+1}e^{-n}\right)$ (b) $\sum\left(\frac{(-1)^{n+1}}{\cosh(n)}\right)$ (c) $\sum\left(\frac{(-1)^{n+1}n}{n^2+1}\right)$

(d) $\sum\left(\frac{(-1)^{n+1}n}{4n^2-9}\right)$ (e) $\sum\left((-1)^{n+1}n^4e^{-n}\right)$

(f) $\sum\left(\frac{(-1)^{n+1}\sqrt{n}}{n+1}\right)$ (g) $\sum\left(\frac{(-1)^{n+1}}{n^{1/p}}\right)$ where $p>0$

(h) $\sum\left(\frac{(-1)^{n+1}}{n^{1/n}}\right)$

4. Discuss the convergence or divergence of the following series:

   (a) $\sum_{n=2}^{\infty}\left(\frac{(-1)^{n+1}}{[\ln(n)]^{p}}\right)$ where $p>0$ (b) $\sum\left((-1)^{n+1}\frac{(n+1)^{n}}{n^{n}}\right)$

5. Prove that if $f(x)$ has a derivative $f'(x)$ such that $\forall x>0$
$$f'(x)<0$$
then $\forall n\geq 1$
$$f(n+1)<f(n)$$

6. Prove that if $f(x)$ has a derivative $f'(x)$ such that $\forall x>0$
$$f'(x)>0$$
then $\forall n\geq 1$
$$\frac{1}{f(n+1)}<\frac{1}{f(n)}$$

7. Give an example of a real positive sequence $(a_n)$ such that $\forall n\in\mathbb{N}$ $a_{n+1}<a_n$ but $\lim_{n\to\infty}(a_n)\neq 0$.

8. Prove corollary (2.19).

9. Let $a_n\in\mathbb{R}^{+}$. Prove that if $(a_n)$ is a decreasing sequence and $\lim_{n\to\infty}(a_n)=0$ then the series $\sum(a_n\cos(n\pi))$ converges.

10. Prove that if $\sum\left((-1)^{n+1}a_n\right)$ converges then
$$\sum\left((-1)^{n+1}a_n\right)=-\sum\left((-1)^{n}a_n\right)$$

## Solutions

1. (a) Converges (b) Converges (c) Converges
   (d) Converges
   (e) Diverges because $\lim_{n\to\infty}\left(\frac{(-1)^{n+1}n}{n+1}\right)$ does not exist.
   (f) Converges, actually $\sum\left(\frac{(-1)^{n+1}}{2n-1}\right)=\tan^{-1}(1)=\frac{\pi}{4}$
   (g) Diverges because $\lim_{n\to\infty}(n!)$ is **not** zero.
   (h) Diverges because $\lim_{n\to\infty}\left(\frac{(-1)^{n+1}n^2}{n^2+1}\right)$ does **not** exist.
   (i) Converges
   (j) Converges. Condition 3) is shown by
   $$a_{n+1}=\frac{n+1}{2^{n+1}}=\frac{1}{2^n}\left(\frac{n+1}{2}\right)<\frac{1}{2^n}(n)=a_n$$
   (k) Converges. Condition 3) is shown by
   $$a_{n+1}=\frac{1}{(n+1)\ln(n+1)}<\frac{1}{n\ln(n)}=a_n$$
2. (a) $f'(x)=e^x>0 \qquad \forall x\in\mathbb{R}$
   (b) $f'(x)=-e^{-x}<0 \qquad \forall x\in\mathbb{R}$
   (c) $f'(x)=\cos(x)<0 \qquad x\in\left]\frac{\pi}{2},\frac{3\pi}{2}\right[$
   (d) $f'(x)=-\sin(x)>0 \qquad x\in\left]\pi,2\pi\right[$
   (e) $f'(x)=-\frac{1}{x^2}<0$ for $x\neq 0$
   (f) $f'(x)=\frac{1-x^2}{(x^2+1)^2}<0$ for $x>1$
   (g) $f'(x)=\sinh(x)>0$ for $x>0$
   (h) $f'(x)=e^x(1+x)>0$ for $x>-1$
3. Use the increasing decreasing theorem (2.18) in each case.
   (a) Use Question 2(b)
   (b) Consider $f(x)=\frac{1}{\cosh(x)}$ and then show that the derivative is negative.
   (c) Consider $f(x)=\frac{x}{x^2+1}$.
   (d) Consider $f(x)=\frac{x}{4x^2-9}$ then $f'(x)=-\frac{(4x^2+9)}{(4x^2-9)^2}<0$

(e) To show that $\lim_{n\to\infty}(a_n)=0$ you need to make repetitive application of L' Hopital's rule. Also consider

$$f(x)=x^4e^{-x} \quad \text{then} \quad f'(x)=x^3e^{-x}(4-x)<0 \quad \text{for } x>4$$

(f) Consider the function $f(x)=\frac{\sqrt{x}}{x+1}$ then $f'(x)=\frac{1-x}{2\sqrt{x}(x+1)^2}<0$ for $x>1$.

(g) Consider $f(x)=\frac{1}{x^{1/p}}$ then $f'(x)=-\frac{1}{p}\left(x^{-1-\frac{1}{p}}\right)<0$ for $x>0$.

(h) Use the above (g) with $p=n$.

4. (a) Converges. (b) Diverges because $\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n=e$
5. Negative derivative means the function is decreasing.
6. Let $g(x)=\frac{1}{f(x)}$ then show that $g'(x)<0$.
7. Let $a_n=1+\frac{1}{n}$ is one example.
8. We have a finite series and a convergent series so therefore we have convergence.
9. Note that $\cos(n\pi)=(-1)^n$ so we have alternating series.
10. Write out the series in the expanded form.