Exercise 2i

1. Determine the interval and radius of convergence of the following power series:

   (a) $\sum_{n=0}^{\infty}\left(x^n\right)$ (b) $\sum_{n=1}^{\infty}\left(\frac{x^n}{2n}\right)$ (c) $\sum_{n=1}^{\infty}\left(\frac{x^n}{n^2}\right)$

   (d) $\sum_{n=1}^{\infty}\left(\frac{nx^n}{2n+1}\right)$ (e) $\sum_{n=0}^{\infty}\left(\frac{x}{\sqrt{2}}\right)^n$ (f) $\sum_{n=0}^{\infty}\left(\frac{n^2}{2^n}\right)x^n$

   (g) $\sum_{n=0}^{\infty}(kn+1)x^n$ where k is an integer constant

2. Determine the interval of convergence of :

   (a) $\sum_{n=0}^{\infty}\left(\frac{x^n}{n!}\right)$ (b) $\sum_{n=1}^{\infty}\left(\frac{(-1)^{n-1}x^n}{n}\right)$ (c) $\sum_{n=0}^{\infty}\left(\frac{(-1)^n x^{2n}}{(2n)!}\right)$

   (d) $\sum_{n=0}^{\infty}\left(\frac{(-1)^n x^{2n+1}}{(2n+1)}\right)$ (e) $\sum_{n=0}^{\infty}\left(x^{2n}\right)$ (f) $\sum_{n=0}^{\infty}\left(x^{n^2}\right)$

3. (a) $\sum_{n=1}^{\infty}\left(\frac{(x+1)^n}{n}\right)$ (b) $\sum_{n=1}^{\infty}\left(\frac{(x-2)^n}{n^2}\right)$ (c) $\sum_{n=0}^{\infty}\left(\frac{(-1)^n(x-1)^n}{2^n}\right)$

   (d) $\sum_{n=0}^{\infty}\left(\frac{n}{2n+1}\left(\frac{x+3}{2}\right)^n\right)$ (e) $\sum_{n=1}^{\infty}\left(\frac{(-1)^n(x-b)^n}{n}\right)$

4. Show that $L=1$ for each of the series of part (III) of (2.22):

   (a) $\sum\left(\frac{1}{n}\right)$ (b) $\sum\left(\frac{(-1)^{n+1}}{n^2}\right)$ (c) $\sum\left(\frac{(-1)^{n+1}}{n}\right)$

5. Find the interval of convergence of:

   (a) $\sum_{n=0}^{\infty}\left(nx^{n-1}\right)$ (b) $\sum_{n=1}^{\infty}\left(\frac{(-1)^n x^n}{\sqrt{n}}\right)$ (c) $\sum_{n=0}^{\infty}\left(n^p x^n\right)$

   (d) $\sum_{n=1}^{\infty}\left(\frac{x^n}{n^2\sqrt{n}}\right)$ (e) $\sum_{n=1}^{\infty}\left(\frac{(-3)^n x^n}{n\sqrt{n}}\right)$ (f) $\sum_{n=0}^{\infty}\left(x^n n^2 e^{-2n}\right)$

   (g) $\sum_{n=0}^{\infty}\left(c^{n^2}x^n\right)$ (h) $\sum_{n=0}^{\infty}\left(\frac{n^n}{(n!)^2}x^n\right)$ (i) $\sum_{n=0}^{\infty}\left(\frac{n^n}{n!}x^n\right)$

6. Discuss the radius of convergence, R, for the cases $R=0$ and $R=+\infty$ for a general power series $\sum_{n=0}^{\infty}\left(c_n x^n\right)$.

7. Let the power series $\sum_{n=0}^{\infty}\left(c_n x^n\right)$ and $\sum_{n=0}^{\infty}\left(d_n x^n\right)$ have the same radius of convergence, R. Show that the radius of convergence of

$$\sum_{n=0}^{\infty}\left(c_n + d_n\right)x^n$$

is also R.

8. Let the power series $\sum_{n=0}^{\infty}\left(c_n x^n\right)$ have the radius of convergence, R. Show that for any non-zero constant k the power series

$$\sum_{n=0}^{\infty}\left(kc_n x^n\right)$$

has the same radius of convergence, R.

9. Let the power series $\sum_{n=0}^{\infty}\left(c_n x^n\right)$ and $\sum_{n=0}^{\infty}\left(d_n x^n\right)$ have the radius of convergence R_1 and R_2 respectively. Show that the power series $\sum_{n=0}^{\infty}\left(c_n + d_n\right)x^n$ has the radius of convergence R where $R = \min\left\{R_1,\ R_2\right\}$.

10. Prove that the radius of convergence of the power series $\sum_{n=0}^{\infty}\left(c_n x^n\right)$ and $\sum_{n=0}^{\infty}\left(c_n\left(x-b\right)^n\right)$ are equal.

11. Prove that the radius of convergence of the power series $\sum_{n=0}^{\infty}\left(c_n x^n\right)$ and $\sum_{n=0}^{\infty}\left(nc_n x^{n-1}\right)$ are equal.

12. Let $\left(a_n\right)$ be the Fibonacci sequence that is

$$a_1 = a_2 = 1 \quad \text{and} \quad a_{n+1} = a_n + a_{n-1}$$

(i) By using induction, or otherwise, prove that $\frac{a_{n+1}}{a_n} \le 2$.

(ii) Prove that the power series $\sum\left(a_n x^n\right)$ converges absolutely for $-\frac{1}{2} < x < \frac{1}{2}$.

Solutions

1. (a) Interval of (absolute) convergence is $-1 < x < 1$. Radius of convergence is $R = 1$.
   (b) Interval of convergence is $-1 \le x < 1$. Radius of convergence is $R = 1$.
   (c) Interval of (absolute) convergence is $-1 \le x \le 1$. Radius of convergence is $R = 1$.
   (d) Interval of (absolute) convergence is $-1 < x < 1$. Radius of convergence is $R = 1$.
   (e) Interval of (absolute) convergence is $-\sqrt{2} < x < \sqrt{2}$. Radius of convergence is $R = \sqrt{2}$.
   (f) Interval of (absolute) convergence is $-2 < x < 2$. Radius of convergence is $R = 2$.
   (g) Interval of (absolute) convergence is $-1 < x < 1$. Radius of convergence is $R = 1$.
2. Interval of convergence is
   (a) $-\infty < x < +\infty$ (absolutely) (b) $-1 < x \le 1$
   (c) $-\infty < x < +\infty$ (absolutely) (d) $-1 \le x \le 1$
   (e) $-1 < x < 1$ (absolutely) (f) $-1 < x < 1$ (absolutely)
3. Interval of convergence is
   (a) $-2 \le x < 0$ (b) $1 \le x \le 3$ (absolutely)
   (c) $-1 < x < 3$ (absolutely) (d) $-5 < x < -1$ (absolutely)
   (e) $b-1 < x \le b+1$
4. (a) $\lim_{n\to\infty}\left(\frac{n}{n+1}\right) = 1$ (b) $\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 = 1$ (c) $\lim_{n\to\infty}\left(\frac{n}{n+1}\right) = 1$
5. Interval of convergence is
   (a) $-1 < x < 1$ (absolutely) (b) $-1 < x \le 1$ (c) $-1 < x < 1$ (absolutely)
   (d) $-1 \le x \le 1$ (absolutely) (e) $-\frac{1}{3} \le x \le \frac{1}{3}$ (absolutely)
   (f) $-e^2 < x < e^2$ (absolutely) (g) $-\infty < x < +\infty$ (absolutely)
   (h) $-\infty < x < +\infty$ (absolutely) (i) $-\frac{1}{e} \le x \le \frac{1}{e}$ (absolutely)