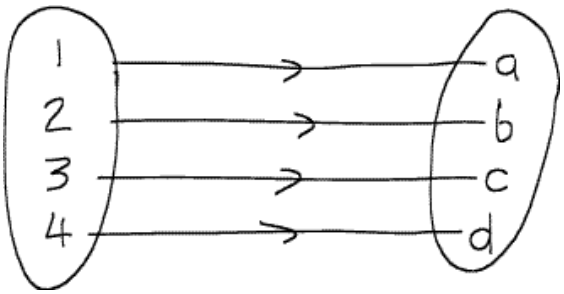
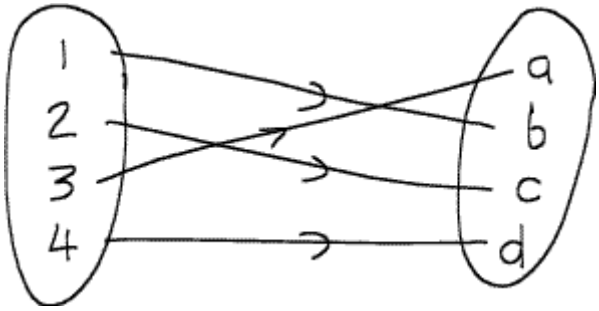
Exercise **3(a)**

1. Let $A=\{1,\ 2,\ 3,\ 4\}$ and $B=\{a,\ b,\ c,\ d\}$. Let $f:A\to B$ be defined by the following diagrams. In each case decide whether f is a function or not. If f is not a function then explain why not.

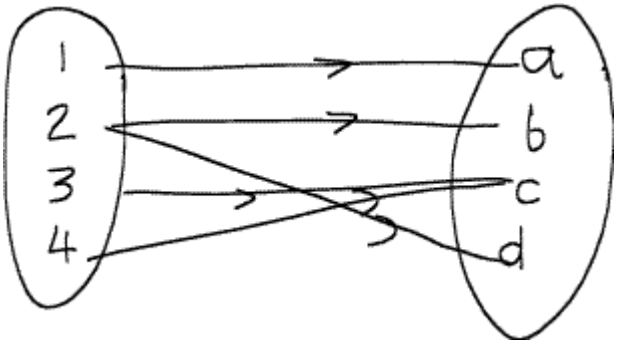
(a)



(b)



(c)



(d)

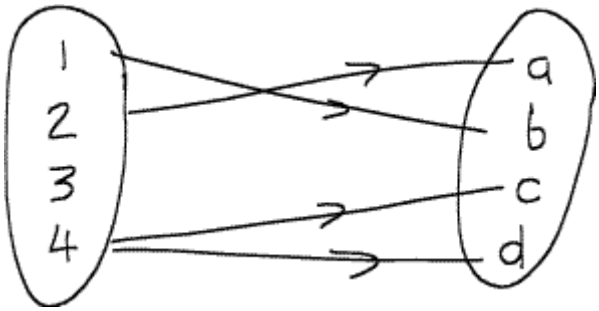


Fig 11

2. Let $f:\mathbb{R}\to\mathbb{R}$ be given by

$$f(x)=x^2+x$$

Find the images of the following under the function f.

(a) 2 (b) 7 (c) y (d) $(a+b)$

3. Consider Dirichlet's function

$$f(x)=\begin{cases}0 & \text{if } x \text{ is irrational}\\ 1 & \text{if } x \text{ is rational}\end{cases}$$

Determine the images of the following:

(a) π (b) e (c) 1 (d) $\sqrt{3}$

(e) $\frac{2}{3}$ (f) $\sqrt{4}$ (g) $1+\sqrt{5}$ (h) 1.414

4. Let $f:\mathbb{R}\to\mathbb{R}$ be given:

$$f(x)=\sqrt{x}$$

The same as in Example 6. Redefine the codomain so that this is a function.

5. Let $f:\mathbb{N}/\{1\}\to\mathbb{Z}$ be given by

$$f(n)=\begin{cases}n & \text{if } n \text{ is prime}\\ -n & \text{if } n \text{ is composite}\end{cases}$$

Determine the following:

(a) $f(13)$ (b) $f(33)$ (c) $f\left(\frac{1}{2}\right)$

What is the range of f ?

(Remember a composite number is a positive integer that is not prime).

6. Determine the domain, codomain and range for each of the following functions:

Let $A=\{-3,\ -1,\ 0,\ 1,\ 3\}$ and $f:A\to\mathbb{R}$ be given by

(a) $f(x)=x^2-1$ (b) $f(x)=x^2+1$ (c) $f(x)=x^2+5x-2$

7. Determine the range of the following functions.

Let $A=\left\{-\frac{\pi}{2},\ -\frac{\pi}{4},\ 0,\ \frac{\pi}{4},\ \frac{\pi}{2}\right\}$ and $g:A\to\mathbb{R}$ be given by

(a) $g(x)=\cos(x)$ (b) $g(x)=\sin(x)$ (c) $g(x)=\cos^2(x)+\sin^2(x)$

8. Let $f:\mathbb{R}\to\mathbb{R}$ be a function given by $f(x)=x^2+3x-2$. Determine

(a) $f(y)$ (b) $f\left(\sqrt{2}\right)$ (c) $f(a+b)$

9. Let $g:\mathbb{R}^+\to\mathbb{R}$. Determine $g(x+y)$ for each of the following functions:

(a) $g(x)=e^x$ (b) $g(x)=\ln(x)$ (c) $\sin(x)$ (d) $\cos(x)$

10. The factorial function $f:\mathbb{N}\to\mathbb{N}$ is defined by

$$f(n)=1\times 2\times 3\times 4\times\ldots\times(n-1)\times n$$

Determine $f(5)$, $f(7)$, $f(10)$, $f(69)$ and $f\left(\frac{1}{2}\right)$.

11. The tau function $\tau : \mathbb{N} \to \mathbb{N}$ is defined by

$$\tau(n) = \text{The number of positive divisors of } n$$

For example $\tau(6) = 4$ because the following 4 positive integers - 1, 2, 3 and 6 divide into 6 exactly.
Determine $\tau(10)$, $\tau(12)$, $\tau(20)$, $\tau(25)$ and $\tau(40)$.

12. The signum function $\text{sgn} : \mathbb{R} \to \mathbb{R}$ is defined by

$$\text{sgn}(x) = \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Determine the domain, codomain and range of $\text{sgn}(x)$.

13. Let $g : \mathbb{N} \to \mathbb{N}$ be given by $g(n) = 2^n$. Determine the range of g. Show that
(i) $g(n+1) = 2g(n)$ (ii) $g(n+2) = 4g(n)$ (iii) $g(n+3) = 8g(n)$
Prove that $g(n+m) = 2^m g(n)$.

14. Let $g : \mathbb{Z} \to \mathbb{Q}$ be given by $g(n) = 2^n$. Determine the range of g. Prove that

$$g(n+m) = 2^m g(n)$$

Some Solutions to Exercise 3a

4. Redefine the codomain to be the set of complex numbers, $\mathbb{C}$. Hence we have the function $g : \mathbb{R} \to \mathbb{C}$ given by $g(x) = \sqrt{x}$.

5. (a) $f(13) = 13$ (b) $f(33) = -33$ (c) $f\left(\frac{1}{2}\right)$ is not defined because ½ is not in the domain. The domain is all the natural numbers apart from 1.
The range is all the integers apart from 0, 1and -1. Hence this can be written as $\mathbb{Z} / \{-1,\ 0,\ 1\}$.