

Exercise 3(d)

Attempt questions 1 to 9 inclusive in Exercise 3(d) of Engineering Mathematics through Applications on pages 154 to 155. In addition try the questions below.

Workbook Questions written in bold. (Questions 12 and 17).

10. Let $f(x) = x + 1$ and $g(x) = x^2$ where x is a real number.

If the domain in each case is the interval $[-1, 1]$ then find the range of

(a) $(f \circ g)(x)$ (b) $(g \circ f)(x)$

(An interval $[a, b]$ is all the real numbers x such that $a \leq x \leq b$).

11. Determine which of the following functions are equal:

(a) $f: \mathbb{N} \rightarrow \mathbb{N}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ be functions given by

$$f(x) = x + 1 \quad \text{and} \quad g(y) = y + 1$$

(b) $f: \{1\} \rightarrow \mathbb{N}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ be functions given by

$$f(x) = x - 1 \quad \text{and} \quad g(x) = 0$$

(c) $f: \{1, 2\} \rightarrow \{2, 4\}$ and $g: \{1, 2\} \rightarrow \{2, 4\}$ be functions given by

$$f(x) = 2x \quad \text{and} \quad g(x) = x^2 - x + 2$$

(d) $f: [0, 2\pi] \rightarrow [-1, 1]$ and $g: [0, 2\pi] \rightarrow [-1, 1]$ be functions given by

$$f(x) = \sin(x) \quad \text{and} \quad g(x) = \cos\left(x - \frac{\pi}{2}\right)$$

12. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ be functions defined by

$$f(x) = 2x - 3 \quad \text{and} \quad g(x) = x^3$$

(a) **Establish that both f and g are bijective and so have an inverse.**

(b) **Determine f^{-1} and g^{-1} .**

(c) **Determine $g^{-1} \circ f^{-1}$ and $f^{-1} \circ g^{-1}$**

(d) **Find $f \circ g$ and $g \circ f$**

What do you notice about your results?

13. Let $f: \mathbb{N} \rightarrow \mathbb{Q}$ and $g: \mathbb{N} \rightarrow \mathbb{N}$ be given by

$$f(n) = \frac{1}{n} \quad \text{and} \quad g(n) = n^2$$

Determine the range of

(a) $(f \circ g)(n)$ (b) $(g \circ f)(n)$ (c) $(g \circ f)\left(\frac{1}{n}\right)$

14. Let $f: \mathbb{R}/\{0\} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ be functions defined by

$$f(x) = \frac{1}{x} \quad \text{and} \quad g(x) = x^2 + 1$$

Determine the range of

$$(a) (f \circ g)(x) \qquad (b) (g \circ f)(x)$$

15. Let $f: \mathbb{R} \rightarrow \{0, 1\}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ be functions defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases} \quad \text{and} \quad g(x) = x$$

Determine

$$(a) (f \circ g)(x) \qquad (b) (g \circ f)(x)$$

16. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R}^+ \rightarrow \mathbb{R}$ be functions defined by

$$f(x) = x - 3 \quad \text{and} \quad g(x) = \sqrt{x}$$

Determine

$$(a) (f \circ g)(x) \qquad (b) (g \circ f)(x)$$

17. Let $f: \mathbb{R}/\{0\} \rightarrow \mathbb{R}$ be a function defined by $f(x) = \frac{1}{x}$. Determine

$$f \circ f \circ f \circ f \circ f.$$

18. Let $f: \mathbb{R} \rightarrow A$ where $A = \{x \mid x \in \mathbb{R} \text{ and } -1 < x < 1\}$ and

$$f(x) = \frac{x}{\sqrt{x^2 + 1}}$$

Show that f is bijective and find f^{-1} . Also determine $(f \circ f^{-1})(x)$ and $(f^{-1} \circ f)(x)$. What do you notice about your results?