Exercise 5a

1. Determine the real numbers $\sup(S)$ and $\inf(S)$ for the following sets S. In each case decide whether $\sup(S)$ and $\inf(S)$ are members of the set S.

(a) $S=\{x \mid x>1 \text{ and } x\in\mathbb{R}\}$

(b) $S=\{x \mid x\le 2 \text{ and } x\in\mathbb{R}\}$

(c) $S=\{x \mid 1\le x\le 2 \text{ and } x\in\mathbb{R}\}$

(d) $S=\{x \mid 1< x< 2 \text{ and } x\in\mathbb{R}\}$

(e) $S=\left\{\cos(x) \,\middle|\, 0<x<\frac{\pi}{2} \text{ and } x\in\mathbb{R}\right\}$

(f) $S=\left\{\sin(x) \,\middle|\, 0\le x<\frac{\pi}{4} \text{ and } x\in\mathbb{R}\right\}$

(g) $S=\{e^{x} \mid x\ge 0 \text{ and } x\in\mathbb{R}\}$

(h) $S=\{e^{-x} \mid x>0 \text{ and } x\in\mathbb{R}\}$

(i) $S=\{\ln(x) \mid x>1 \text{ and } x\in\mathbb{R}\}$

(j) $S=\left\{\frac{1}{x} \,\middle|\, x\ge 1 \text{ and } x\in\mathbb{R}\right\}$

(k) $S=\left\{\frac{x-1}{x} \,\middle|\, x>1 \text{ and } x\in\mathbb{R}\right\}$

2. Determine the real numbers $\sup(S)$ and $\inf(S)$ for the following sets S. Also find whether $\sup(S)$ and $\inf(S)$ are members of the set S.

(a) $S=\{n \mid n\in\mathbb{N} \text{ and } n\ge 2\}$

(b) $S=\left\{\frac{1}{n} \,\middle|\, n\in\mathbb{N} \text{ and } n\ge 2\right\}$

(c) $S=\left\{\frac{1}{n} \,\middle|\, n\in\mathbb{Z} \text{ and } n\ne 0\right\}$

(d) $S=\left\{\frac{1}{2^{n}} \,\middle|\, n\in\mathbb{N}\right\}$

(e) $S=\left\{\frac{1}{2^{n}} \,\middle|\, n\in\mathbb{N} \text{ and } n=0\right\}$

(f) $S=\{2^{n} \mid n\in\mathbb{N}\}$

(g) $S=\left\{\frac{1}{10^{n}} \,\middle|\, n\in\mathbb{N}\right\}$

(h) $S=\left\{(-1)^{n}+\frac{1}{n} \,\middle|\, n\in\mathbb{N}\right\}$

(i) $S = \left\{ 1 + \frac{(-1)^n}{n} \,\middle|\, n \in \mathbb{N} \right\}$

(j) $S = \left\{ 1 - \frac{(-1)^n}{n} \,\middle|\, n \in \mathbb{N} \right\}$

(k) $S = \left\{ \frac{1}{2^n} - \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\}$

3. Give an example of a set A with $\inf(A) = \pi$ and $\pi \in A$.
4. Give an example of a set which is unbounded.
5. Give an example of a set S such that $\sup(S) = \inf(S)$.

Solutions

1. (a) $\inf(S) = 1$, $\sup(S)$ does **not** exist, $\inf(S) \notin S$.

   (b) $\sup(S) = 2$, $\inf(S)$ does **not** exist, $\sup(S) \in S$.

   (c) $\sup(S) = 2$, $\inf(S) = 1$, $\sup(S) \in S$ and $\inf(S) \in S$

   (d) $\sup(S) = 2$, $\inf(S) = 1$, $\sup(S) \notin S$ and $\inf(S) \notin S$

   (e) $\sup(S) = 1$, $\inf(S) = 0$, $\sup(S) \notin S$ and $\inf(S) \notin S$

   (f) $\sup(S) = \frac{1}{\sqrt{2}}$, $\inf(S) = 0$, $\sup(S) \notin S$ and $\inf(S) \in S$

   (g) $\sup(S)$ does **not** exist, $\inf(S) = 1$, $\inf(S) \in S$

   (h) $\sup(S) = 1$, $\inf(S) = 0$, $\sup(S) \notin S$ and $\inf(S) \notin S$

   (i) $\sup(S)$ does **not** exist, $\inf(S) = \ln(1)$, $\inf(S) \notin S$

   (j) $\sup(S) = 1$, $\inf(S) = 0$, $\sup(S) \notin S$ and $\inf(S) \notin S$

   (k) $\sup(S) = 1$, $\inf(S) = 0$, $\sup(S) \notin S$ and $\inf(S) \notin S$

2. (a) $\sup(S)$ does **not** exist, $\inf(S) = 1$, $\inf(S) \in S$

   (b) $\sup(S) = \frac{1}{2}$, $\inf(S) = 0$, $\sup(S) \in S$ and $\inf(S) \notin S$

   (c) $\sup(S) = 1$, $\inf(S) = -1$, $\sup(S) \in S$ and $\inf(S) \in S$

   (d) $\sup(S) = \frac{1}{2}$, $\inf(S) = 0$, $\sup(S) \in S$ and $\inf(S) \notin S$

   (e) $\sup(S) = 1$, $\inf(S) = 0$, $\sup(S) \in S$ and $\inf(S) \notin S$

   (f) $\sup(S)$ does **not** exist, $\inf(S) = 2$, $\inf(S) \in S$

   (g) $\sup(S) = \frac{1}{10}$, $\inf(S) = 0$, $\sup(S) \in S$ and $\inf(S) \notin S$

   (h) $\sup(S) = \frac{3}{2}$, $\inf(S) = -1$, $\sup(S) \in S$ and $\inf(S) \notin S$

   (i) $\sup(S) = \frac{3}{2}$, $\inf(S) = 0$, $\sup(S) \in S$ and $\inf(S) \in S$

(j) $\sup(S) = 2$, $\inf(S) = \frac{1}{2}$, $\sup(S) \in S$ and $\inf(S) \in S$

(k) $\sup(S) = 0$, $\inf(S) = -\frac{1}{2}$, $\sup(S) \notin S$ and $\inf(S) \in S$

3. $A = \{x \mid x \geq \pi \text{ and } x \in \mathbb{R}\}$

4. The set of all real numbers, $\{x \mid x \in \mathbb{R}\}$.

5. $S = \{C \mid x \in \mathbb{R}\}$ where C is a constant. Hence $\sup(S) = \inf(S) = C$.

A set with one element will also do such as $S = \{5\}$.