Exercise 5(d)

Workbook Questions 7 and 12 in bold.

1. Prove that if C is a constant then $\lim_{n\to\infty}(C) = C$.

2. Prove that if K is a constant and $\lim_{n\to\infty}(x_n)$ is a real convergent sequence then $\lim_{n\to\infty}(Kx_n) = K\lim_{n\to\infty}(x_n)$.

3. Let (x_n) and (y_n) be real convergent sequences and α and β be constants. Prove that
$$\lim_{n\to\infty}(\alpha x_n + \beta y_n) = \alpha\lim_{n\to\infty}(x_n) + \beta\lim_{n\to\infty}(y_n)$$

4. Let (x_n), (y_n) and (z_n) be real convergent sequences such that
$$\lim_{n\to\infty}(x_n) = L,\ \lim_{n\to\infty}(y_n) = M \text{ and } \lim_{n\to\infty}(z_n) = K$$
Prove that $\lim_{n\to\infty}(x_n + y_n + z_n) = L + M + K$.

5. Let (x_n) be a real convergent sequence such that $\lim_{n\to\infty}(x_n) = L$. Prove that $\lim_{n\to\infty}(-x_n) = -L$

6. Prove Proposition (5.15) part (ii):
Let (x_n) and (y_n) be real convergent sequences such that $\lim_{n\to\infty}(x_n) = L$ and $\lim_{n\to\infty}(y_n) = M$. Then
$$\lim_{n\to\infty}(x_n - y_n) = L - M$$

7. **Prove Proposition (5.15) part (iv):**
Let (x_n) and (y_n) be real convergent sequences such that $\lim_{n\to\infty}(x_n) = L$ and $\lim_{n\to\infty}(y_n) = M$. Then
$$\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right) = \frac{L}{M} \text{ provided that } \forall n \in \mathbb{N}\ \ y_n \neq 0 \text{ and } M \neq 0$$

8. Give an example of a sequence which is bounded but **not** convergent.

9. Let (x_n) be a real convergent sequence such that $\lim_{n\to\infty}(x_n) = L$. Prove that
$$\lim_{n\to\infty}(|x_n|) = |L|$$

10. Let both (x_n) and (y_n) be real convergent sequences. Prove that if for all $n \in \mathbb{N}$ we have the inequality $x_n \le y_n$ then $\lim_{n\to\infty}(x_n) \le \lim_{n\to\infty}(y_n)$.

11. Let (x_n) be a real convergent sequence such that $\lim_{n\to\infty}(x_n) = L$. If for all $n \in \mathbb{N}$ we have $K \le x_n \le M$ where K and M are real numbers then prove that $K \le L \le M$.

12. (**Sandwich Rule**). **Let (x_n), (y_n) and (z_n) be real convergent sequences such that for all $n \in \mathbb{N}$ we have $x_n \le y_n \le z_n$. If $\lim_{n\to\infty}(x_n) = \lim_{n\to\infty}(z_n) = L$ prove that $\lim_{n\to\infty}(y_n) = L$. (This is also called the Squeeze Theorem).**

13. Let (x_n) be a real convergent sequence such that $\lim_{n\to\infty}(x_n) = 0$ and (y_n) be a bounded sequence. Prove that $\lim_{n\to\infty}(x_n y_n) = 0$.

14. Let (x_n) and be a real convergent sequence such that $\lim_{n\to\infty}(x_n) = L$. Prove that

$$\lim_{n\to\infty}\left(x_n^{\ 2}\right) = L^2$$

By
(i) using proposition (5.15)
(ii) using the formal definition of the limit (5.11).

15. Let (x_n) be a real convergent sequence such that $\lim_{n\to\infty}(x_n) = L$. Prove that

$$\lim_{n\to\infty}\left(x_n^{\ m}\right) = L^m \quad \text{where } m \in \mathbb{N}$$

16. Let (x_n) be a real convergent sequence such that $\lim_{n\to\infty}(x_n) = L$ and for all $n \in \mathbb{N}$ $x_n \ge 0$. Prove that

$$\lim_{n\to\infty}\left(\sqrt{x_n}\right) = \sqrt{L}$$

[Hint: $\sqrt{a} - \sqrt{b} = \dfrac{a-b}{\sqrt{a}+\sqrt{b}}$]

17. Let (x_n) be a real convergent sequence such that $\lim_{n\to\infty}(x_n) = L$. Prove that

$$\lim_{n\to\infty}\left(\cos(x_n)\right) = \cos(L)$$

[Hint: For all $x \in \mathbb{R}$ we have the inequality $|\sin(x)| \le |x|$].

Solutions and Hints to Exercise 5(d)

1. $|C - C| = 0 < \varepsilon$
2. Use (5.15) part (iii) $\lim_{n\to\infty}(x_n y_n) = \lim_{n\to\infty}(x_n)\lim_{n\to\infty}(y_n)$ to show $\lim_{n\to\infty}(Kx_n) = K\lim_{n\to\infty}(x_n)$
3. Apply $\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty}(x_n) + \lim_{n\to\infty}(y_n)$.

4. $\left|x_n - L\right| < \dfrac{\varepsilon}{3}$, $\left|y_n - M\right| < \dfrac{\varepsilon}{3}$ and $\left|z_n - K\right| < \dfrac{\varepsilon}{3}$

5. $\left|\left(-x_n\right) - \left(-L\right)\right| = \left|x_n - L\right|$

6. Rewrite $x_n - y_n$ as $x_n + \left(-1\right) y_n$ then $\lim\limits_{n\to\infty}\left(x_n - y_n\right) = \lim\limits_{n\to\infty}\left(x_n + \left(-1\right) y_n\right)$ use result of question 3 on this.

8. $x_n = \left(-1\right)^n$.

9. Use the inequality $\left|\left|x_n\right| - \left|L\right|\right| \le \left|x_n - L\right|$

10. See the proof of proposition (5.17).

11. Use the result of question 10.

13. Since $\left(y_n\right)$ is bounded therefore $\left|y_n\right| \le K$ where $K > 0$. Take $\left|x_n\right| < \dfrac{\varepsilon}{K}$.

14. (i) Use proposition (5.15) part (iii).

(ii) $\left|x_n - L\right| < \dfrac{\varepsilon}{K + \left|L\right|}$ where for all $n \in \mathbb{N}$ $\left|x_n\right| \le K$ and $K > 0$.

15. Use mathematical induction.

$$\lim_{n\to\infty}\left(x_n^{\,k+1}\right) = \lim_{n\to\infty}\left(x_n^{\,k}\right)\lim_{n\to\infty}\left(x_n\right) = L^k L = L^{k+1}$$

16. Consider two cases, $L = 0$ and $L > 0$. For $L > 0$ take $\left|x_n - L\right| < \sqrt{L}\varepsilon$.

17. Use the trigonometric identity:

$$\cos\left(A\right) - \cos\left(B\right) = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

Also need to use $\left|\sin\left(x\right)\right| \le 1$.