Exercise 5(e)

1. Determine the following limits:

(a) $\lim_{n\to\infty}\left(\frac{n-1}{n+1}\right)$ (b) $\lim_{n\to\infty}\left(\frac{n-5}{5n+1}\right)$ (c) $\lim_{n\to\infty}\left(\frac{3n+3}{2n-1}\right)$

(d) $\lim_{n\to\infty}\left(\frac{\frac{1}{2}n+3}{n-5}\right)$ (e) $\lim_{n\to\infty}\left(\frac{\frac{1}{2}n+3}{\frac{1}{5}n-5}\right)$ (f) $\lim_{n\to\infty}\left(\frac{\frac{1}{3}n}{\frac{1}{15}n}\right)$

2. Evaluate the following limits:

(a) $\lim_{n\to\infty}\left(\frac{2n^2-3}{5n^2+1}\right)$ (b) $\lim_{n\to\infty}\left(\frac{7n^2-2n+5}{3n^2+n+4}\right)$ (c) $\lim_{n\to\infty}\left(\frac{5n^2-n+11}{2n^2+45n-62}\right)$

(d) $\lim_{n\to\infty}\left(\frac{2n^2-2n+1}{3n^4-2n+1}\right)$ (e) $\lim_{n\to\infty}\left(\frac{\frac{1}{3}n^4+1}{7n^4-1}\right)$ (f) $\lim_{n\to\infty}\left(\frac{\frac{9}{2}n^2-n^2+13}{\frac{2}{3}n^2+n-4}\right)$

3. Find the following limits:

(a) $\lim_{n\to\infty}\left(\frac{1}{7^n}\right)$ (b) $\lim_{n\to\infty}\left(\left(\frac{1}{6n}\right)^n\right)$ (c) $\lim_{n\to\infty}\left(\frac{1}{n^n}\right)$

(d) $\lim_{n\to\infty}\left(\frac{4^n-1}{5^n+1}\right)$ (e) $\lim_{n\to\infty}\left(\frac{7^n-n^7+7n}{15^n+n+1}\right)$ (f) $\lim_{n\to\infty}\left(\frac{\frac{1}{5^n}}{\frac{1}{3^n}}\right)$

4. Evaluate the following limits (capital letters are constants):

(a) $\lim_{n\to\infty}\left(\frac{An^2+Bn+C}{Dn^2-En+F}\right)$ where A and D are non zero constants.

(b) $\lim_{n\to\infty}\left(\frac{Cn^4-2}{Dn^4-2n+1}\right)$ where C and D are non zero constants.

(c) $\lim_{n\to\infty}\left(\frac{A^n-n^7+7n}{B^n+n+1}\right)$ where the constants $B>A>1$.

5. Determine the following limits:

(a) $\lim_{n\to\infty}\left(\left(\frac{n}{n+1}\right)^3\right)$ (b) $\lim_{n\to\infty}\left(\left(\frac{n^2-2n+1}{2n^2+1}\right)^3\right)$

(c) $\lim_{n\to\infty}\left(\sqrt{\left(\frac{n^4-n^2+1}{7n^4+1}\right)}\right)$ (*) (d) $\lim_{n\to\infty}\left(\frac{n-1}{\sqrt{3n^2+1}}\right)$

6. Determine $\lim_{n\to\infty}\left(\cos^n(x)\right)$ where $x\neq k\pi \quad (k\in\mathbb{Z})$.

7. Find $\lim_{n\to\infty}\left(\frac{\sin(n)}{n}\right)$.

8. Prove the following results:
If $\lim_{n\to\infty}(x_n)=0$ and $\lim_{n\to\infty}(y_n)=0$ then
(i) $\lim_{n\to\infty}(x_n+y_n)=0$
(ii) $\lim_{n\to\infty}(x_n y_n)=0$
(iii) $\lim_{n\to\infty}(Kx_n)=0$ where K is a constant

9. Prove that

$$\lim_{n\to\infty}(x_n)=0 \quad \Leftrightarrow \quad \lim_{n\to\infty}(|x_n|)=0$$

10. Prove that if $\lim_{n\to\infty}(x_n)=0$ and for all $n\in\mathbb{N}$ we have $x_n\geq 0$ then

$$\lim_{n\to\infty}\left((x_n)^p\right)=0 \text{ where } p>0$$

11. * Find $\lim_{n\to\infty}\left(\sqrt{n-1}-\sqrt{n}\right)$

12. *Prove result (5.20) that is

$$\lim_{n\to\infty}\left(n^r x^n\right)=0 \text{ where } r \text{ is a real number and } |x|<1$$

Brief Solutions to Exercise 5(e)

1. (a) 1	(b) $\frac{1}{5}$	(c) $\frac{3}{2}$	(d) $\frac{1}{2}$	(e) $\frac{5}{2}$	(f) 5
2. (a) $\frac{2}{5}$	(b) $\frac{7}{3}$	(c) $\frac{5}{2}$	(d) 0	(e) $\frac{1}{21}$	(f) $\frac{21}{4}$
3. (a) 0	(b) 0	(c) 0	(d) 0	(e) 0	(f) 0
4. (a) $\frac{A}{D}$		(b) $\frac{C}{D}$	(c) 0		
5. (a) 1		(b) $\frac{1}{8}$	(c) $\frac{1}{\sqrt{7}}$	(d) $\frac{1}{\sqrt{3}}$	

6. The limiting value is 0.
7. The limiting value is 0.
8. (i) Use $\lim_{n\to\infty}(x_n+y_n)=\lim_{n\to\infty}(x_n)+\lim_{n\to\infty}(y_n)$. Similarly for parts (ii) and (iii).
9. ($\Rightarrow$) Assume $\lim_{n\to\infty}(x_n)=0$ prove $\lim_{n\to\infty}(|x_n|)=0$. ($\Leftarrow$) Then assume $\lim_{n\to\infty}(|x_n|)=0$ prove $\lim_{n\to\infty}(x_n)=0$.
10. Use the $\varepsilon-N_0$ definition of the convergence of a sequence. Show $\left|x_n{}^p-0\right|<\varepsilon$.
11. The limiting value is 0. Multiply by $\frac{\sqrt{n-1}+n}{\sqrt{n-1}+n}$.
12. See complete solutions.