Exercise 5(f)

1. By using the "Monotone Convergence Theorem" show that $\lim_{n\to\infty}(x_n)=0$ where $x_n=\frac{1}{n}$.

2. By using the "Monotone Convergence Theorem" show that $\lim_{n\to\infty}(x_n)=0$ where $x_n=\frac{1}{\sqrt{n}}$.

3. Let $x_1=1$ and $x_{n+1}=\frac{1}{5}(x_n+3)$. Prove that $\lim_{n\to\infty}(x_n)=\frac{3}{4}$.

4. Prove that the sequence $x_n=\sqrt{1-\frac{1}{n}}$ is convergent by using the Monotone Convergence Theorem.

5. Let (x_n) be a real sequence defined by

$$x_1=2 \text{ and } x_{n+1}=\frac{x_n}{2}+\frac{3}{2x_n} \qquad (☺)$$

   Prove that the sequence (x_n) defined in (☺) is monotone and bounded. Also prove that $\lim_{n\to\infty}(x_n)=\sqrt{3}$.

6. Let (x_n) be a real sequence defined by

$$x_1=1 \text{ and } x_{n+1}=\sqrt{1+x_n}$$

   Show that the sequence (x_n) is monotone and bounded. Also determine $\lim_{n\to\infty}(x_n)$.

7. Let (x_n) be a real sequence defined by

$$x_1=1 \text{ and } x_{n+1}=\sqrt{2+x_n}$$

   Show that the sequence (x_n) is monotone and bounded. Also determine $\lim_{n\to\infty}(x_n)$.

8. Prove the Monotone Convergence Theorem (5.23) which says- "A **bounded monotone** sequence of real numbers is **convergent**" for a **bounded decreasing** sequence.

9. Prove Proposition (5.25) which says that if (x_n) is a bounded decreasing sequence of real numbers then

$$\lim_{n\to\infty}(x_n)=\inf\{x_n \mid n\in\mathbb{N}\}$$

where inf is the infimum (Greatest Lower Bound) of the set.

Brief Solutions to Exercise 5(f)

1. Show that $x_n = \frac{1}{n}$ is decreasing and then by induction prove $x_n \le 1$.
2. Prove that $x_n = \frac{1}{\sqrt{n}}$ is decreasing by using

$$n+1 > n \quad \Leftrightarrow \quad \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$$

   Then by induction show that $x_n \le 1$.
3. Show by induction that x_n is decreasing by proving

$$x_{k+1} - x_k \le 0$$

   Again by induction show $x_n \ge 3/4$.
4. Show that $x_n = \sqrt{1 - \frac{1}{n}}$ is an increasing sequence and also prove that $x_n \le 1$.
5. Very similar to Example 27. Prove that $x_n \ge \sqrt{3}$ and (x_n) is decreasing.
6. Show that the sequence (x_n) is increasing and it is bounded by proving $x_n \le 1.7$. The limiting value is $\lim_{n\to\infty}(x_n) = 1.618$.
7. Show that the sequence (x_n) is increasing and it is bounded by proving $x_n \le 2$. The limiting value is $\lim_{n\to\infty}(x_n) = 2$.
8. Very similar to proof of (5.23) but use proposition (5.6) rather than (5.5).
9. Proof in solution to Question 8.