Tough Nut to Crack – Integration I: Solution

Show that $\int_0^{\phi} \sec(x)\,dx = \ln\left|\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right|$

Solution by Kuldeep Singh

Let $u=\tan\left(\frac{x}{2}\right)$ then $\frac{du}{dx}=\frac{1}{2}\sec^2\left(\frac{x}{2}\right)$ which gives $\mathrm{d}u=\frac{1}{2}\left(1+u^2\right)dx$

We have the trigonometric identity

$$\cos(x)=\frac{1-u^2}{1+u^2}$$

$$\sec(x)=\frac{1+u^2}{1-u^2}$$

Integrating this last term we have

$$\int \sec(x)\,dx = 2\int \frac{1+u^2}{1-u^2}\,\frac{du}{1+u^2}$$

$$=2\int \frac{du}{1-u^2}$$

$$=2\ln\left|\frac{1+u}{1-u}\right| = 2\ln\left|\frac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}\right|$$

Putting in the limits gives

$$\int_0^{\phi} \sec(x)\,dx = 2\ln\left[\left|\frac{1+\tan(x/2)}{1-\tan(x/2)}\right|\right]_0^{\phi}$$

$$=\ln\left|\frac{1+\tan(\phi/2)}{1-\tan(\phi/2)}\right| = \ln\left|\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right|$$