Chapter 2 : Infinite Series

Section D **Comparison Tests and p-series**

By the end of this section you will be able to

- test a given series for convergence by applying the p-series
- test a given series for convergence by applying the comparison test
- test a given series for convergence by applying the limit comparison test

D1 **Important p-series**

The p-series is an important series because it can be used for a wide range of series and it is easy to apply. It is normally used in conjunction with the comparison tests. The p-series is a simple test for convergence or divergence of a given series.

The p-series (2.11). The series

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^p}\right)\begin{cases}\text{converges if } p>1\\ \text{diverges if } p\le 1\end{cases}$$

Proof. We will prove this result in the section under integral test.

What does the p-series mean in everyday language?

It says the infinite series $\sum_{n=1}^{\infty}\left(\frac{1}{n^p}\right)$ converges if p is greater than 1 but diverges if p is less than or equal to 1. We only need to check the value of p .

What series do we have if $p=1$?

The harmonic series

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$$

which diverges. Let's do some examples.

Example 15

Test $\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)$ for convergence.

Solution.

Can we use the p-series?

Yes because the general term is of the form $\frac{1}{n^p}$. *What is the value of p for the given series?*

Well $p=2$ because we have the general term $\frac{1}{n^2}$. Since $p=2$ which is greater than 1 therefore by the p-series test (2.11), $\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)$ converges.

Example 16

Test $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt[3]{n^2}}\right)$ for convergence.

Solution.

Can we use the p-series test?

Yes we need to rewrite the general term with a numerical index rather than in surd form. *How can we express the given general term* $\frac{1}{\sqrt[3]{n^2}}$ *with a numerical index?*

$$\frac{1}{\sqrt[3]{n^2}} = \frac{1}{n^{2/3}}$$

The value of $p = \frac{2}{3} \leq 1$ therefore by the p-series test

(2.11) $$\sum_{n=1}^{\infty}\left(\frac{1}{n^p}\right) \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}$$

the given series $\sum_{n=1}^{\infty}\left(\frac{1}{n^{2/3}}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{\sqrt[3]{n^2}}\right)$ diverges.

The p-series test is easy to apply because we only need to check if p is less than or equal to 1 or p is greater than 1. *What are the limitations of the p-series?*

1) If the series converges, the p-series does **not** evaluate the sum of the series. The p-series **only** tests whether the series converges or diverges.
2) The test only works if the general term of the series is of the form $\frac{1}{n^p}$. *But how do we test the following series:*

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^p+1}\right),\ \sum_{n=1}^{\infty}\left(\frac{1}{n^p-n}\right),\ \sum_{n=1}^{\infty}\left(\frac{1}{\left[\cos(n)\right]^p}\right),\ \ldots$$

   and many other series like these?
   To test these we will need to use the comparison test in conjunction with the p-series test which is discussed in the next subsections.

Going back to the first bullet point, how can we evaluate the sum of a convergent series?

In general the sum of a convergent infinite series is difficult or even impossible to evaluate. If the given convergent series is geometric or telescoping series then finding the sum is straightforward otherwise it is challenging or impossible. For the remainder of this chapter we will only test a given series for convergence and **not** find the sum of the convergent series. However you can find the sum by using computer algebra systems such as MAPLE, MATHEMATICA, etc

D2 **Comparison Test**

There are two comparison tests:

1) Comparison Test
2) Limit Comparison Test

In this subsection we cover the first test. In general the comparison tests are more challenging and you will need to know your work on inequalities and limits of sequences to be able to apply these tests.

The following inequality is often used for the first comparison test:
Let $x \in \mathbb{R}^+$ and $y \in \mathbb{R}^+$ then

(*) $x \geq y$ implies $\frac{1}{x} \leq \frac{1}{y}$ or $x \leq y$ implies $\frac{1}{x} \geq \frac{1}{y}$

Remember the inequality sign changes if we take the reciprocal.
Also to use the comparison tests we need a stock of series that we know converges or diverges because the comparison test is based on comparing series. Clearly you must know whether one of the series converges or diverges.

Comparison Test (2.12)

If $\forall n \in \mathbb{N}$ (that is for all natural numbers), $0 \leq a_n \leq b_n$ then

(I) $\sum_{k=1}^{\infty}(b_k)$ is convergent $\Rightarrow \sum_{k=1}^{\infty}(a_k)$ is convergent

(II) $\sum_{k=1}^{\infty}(a_k)$ is divergent $\Rightarrow \sum_{k=1}^{\infty}(b_k)$ is divergent

Note: *What does the comparison test mean in everyday language?*

Roughly (I) is saying if the larger series $\sum_{k=1}^{\infty}(b_k)$ converges then the smaller series $\sum_{k=1}^{\infty}(a_k)$ converges. Part (II) says that if the smaller series $\sum_{k=1}^{\infty}(a_k)$ diverges then the larger series $\sum_{k=1}^{\infty}(b_k)$ diverges. Here when we were talking about larger or smaller series we are referring to series containing larger or smaller terms respectively.

Proof of (I). We assume that $\sum_{k=1}^{\infty}(b_k)$ is convergent and try to prove $\sum_{k=1}^{\infty}(a_k)$ converges. Let $B = \sum_{k=1}^{\infty}(b_k)$ and S_n be the nth partial sum of the first n terms of a_k :

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{k=1}^{n}(a_k) \qquad (*)$$

Then the sequence of partial sums (S_n) is bounded by B because

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n \leq \underbrace{b_1 + b_2 + b_3 + \ldots + b_n}_{\text{Larger Terms}} \leq \underbrace{b_1 + b_2 + b_3 + \ldots}_{\text{Infinite Series}} = \sum_{k=1}^{\infty}(b_k) = B$$

Hence for all n we have $S_n \leq B$.

By the definition of partial sum, S_n (see (*) above), and given that for all natural numbers $a_n \geq 0$ therefore (S_n) is an increasing sequence. This means that (S_n) is a monotonic sequence. Since (S_n) is a bounded monotonic sequence it converges by (1.??).

Hence (S_n) converges and so $\lim_{n\to\infty}(S_n) = \sum_{k=1}^{\infty}(a_k)$ converges.

Proof of (II). This is the contrapositive of (I).

■

(1.??) A bounded monotonic sequence converges

Can you remember what the term 'monotonic sequence' means?
Monotonic sequence is an increasing or decreasing sequence.
The symbol k is just a "dummy variable" and we can rewrite the comparison test with a little less notation as:
(2.12) If $0 \le a_n \le b_n$ then

(I) $\sum(b_n)$ is convergent $\Rightarrow$ $\sum(a_n)$ is convergent

(II) $\sum(a_n)$ is divergent $\Rightarrow$ $\sum(b_n)$ is divergent

Let's try applying this comparison test to some examples.

Example 17

Test $\sum_{n=1}^{\infty}\left(\frac{1}{n^2+1}\right)$ for convergence.

Solution.

Which test can we use?

Well it looks close to a p-series but it has n^2+1 rather than just n^2 on the denominator. So let's try using the comparison test (2.12). *How do we apply this test?* We compare the given series with a series we know converges or diverges. *But which one, there are millions of them?*

Well we know $\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)$ series converges because of the p-series test ($p = 2$). The denominator of the general term of the given series is n^2+1 and

$$n^2+1 > n^2$$

$$\frac{1}{n^2+1} < \frac{1}{n^2} \qquad \text{[By (*)]}$$

Since the series with larger terms $\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)$ converges therefore by the comparison test (2.12) part (I) the given series with smaller terms $\sum_{n=1}^{\infty}\left(\frac{1}{n^2+1}\right)$ converges.

The application of the comparison test (2.12) might be confusing because of all the notation involved. In Example 17 the comparison test (2.12) part (I) says that if the series containing the larger terms $b_n = \frac{1}{n^2}$ converges then the series with the smaller terms $a_n = \frac{1}{n^2+1}$ converges.

Also in Example 17 we compared the given series $\sum_{n=1}^{\infty}\left(\frac{1}{n^2+1}\right)$ with a series that we know converges $\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)$. *Why compare with this series?*

(*) $x \ge y$ implies $\frac{1}{x} \le \frac{1}{y}$

(2.12) (I) If $0 \le a_n \le b_n$ $\sum(b_n)$ is convergent implies $\sum(a_n)$ is convergent

We chose to compare with $\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)$ because it is close to the given series $\sum_{n=1}^{\infty}\left(\frac{1}{n^2+1}\right)$ and it is a p-series. Remember you can check the convergence of a p-series by inspection.

As stated earlier we need to have a bank of series that we know converge or diverge.

Example 18

Test $\sum_{n=1}^{\infty}\left(\frac{1}{2^n+1}\right)$ for convergence.

Solution.

We can use the comparison test for the given series $\sum_{n=1}^{\infty}\left(\frac{1}{2^n+1}\right)$. *Why?*

Because a close associate $\sum_{n=1}^{\infty}\left(\frac{1}{2^n}\right)$ is a geometric series and we know this converges because the common ratio $|r|=\frac{1}{2}<1$. Comparing terms we have

$$2^n+1>2^n$$

$$\frac{1}{2^n+1}<\frac{1}{2^n} \qquad \text{[By (*)]}$$

Since the series containing larger terms $\sum_{n=1}^{\infty}\left(\frac{1}{2^n}\right)$ converges therefore by the comparison test (2.12) part (I) the given series with smaller terms $\sum_{n=1}^{\infty}\left(\frac{1}{2^n+1}\right)$ converges.

Again in Example 18 the application of the comparison test (2.12) part (I) says that if the series with the larger terms $b_n=\frac{1}{2^n}$ converges then the series with the smaller terms $a_n=\frac{1}{2^n+1}$ converges.

Also we chose to compare with $\sum\left(\frac{1}{2^n}\right)$ because it is a geometric series and we can identify by inspection whether this converges or not.

The next example is more difficult then the previous ones because we are showing a general result and therefore we need to use symbols like k rather than numbers. The other difficulty is which series do we compare with. An added complexity is that when applying the comparison test, (2.12) part (II), you need to be careful with the inequalities.

(*) $x\ge y$ implies $\frac{1}{x}\le\frac{1}{y}$

(2.12) (I) If $0\le a_n\le b_n$ $\sum(b_n)$ is convergent implies $\sum(a_n)$ is convergent

Example 19

Show that $\sum_{n=1}^{\infty}\left(\frac{1}{kn^p}\right)$ where $p \le 1$ and $k \in \mathbb{N}$ diverges.

Solution.

From our results on inequalities for $p \le 1$ we have

$$kn^p \le kn$$

$$\frac{1}{kn^p} \ge \frac{1}{kn} \qquad \text{[By (*)]}$$

Since $\sum_{n=1}^{\infty}\left(\frac{1}{kn}\right)$ diverges (by question 5 of Exercise 2(b)) therefore by the comparison test (2.12) part (II) the given series with larger or equal terms $\sum_{n=1}^{\infty}\left(\frac{1}{kn^p}\right)$ diverges.

In Example 19 the application of the comparison test (2.12) part (II) means that if the series with the smaller terms $a_n = \frac{1}{kn}$ diverges then the series with the larger terms $b_n = \frac{1}{kn^p}$ diverges. The above is also true if we have equality that is $\frac{1}{kn^p} = \frac{1}{kn}$.

But how are you supposed to remember $\sum\left(\frac{1}{kn}\right)$ *diverges?*

Because $\sum\left(\frac{1}{kn}\right)$ is more or less the harmonic series $\sum\left(\frac{1}{n}\right)$ and we know this diverges. Of course you might say this is **not rigorous** but it is normal practice in these type of questions to have a 'hunch' and then prove the result if you have to.

Clearly we have proven $\sum\left(\frac{1}{kn}\right)$ diverges in Exercise 2b.

Example 20

Test $\sum_{n=1}^{\infty}\left(\frac{1}{5n-1}\right)$ for convergence.

Solution.

Can we use the comparison test (2.12) again?

Yes because the denominator of the general term is $5n-1$ and we know from our work on inequalities that

$$5n-1 < 5n$$

$$\frac{1}{5n-1} > \frac{1}{5n} \qquad \left[\text{By (*)}\right]$$

The series $\sum_{n=1}^{\infty}\left(\frac{1}{5n}\right)$ diverges. *Why?*

This follows from Example 19 above with $k=5$ and $p=1$. Since the series

---

(*) $x \le y$ implies $\frac{1}{x} \ge \frac{1}{y}$

(2.12) (II) If $0 \le a_n \le b_n$ $\sum(a_n)$ is divergent implies $\sum(b_n)$ is divergent containing smaller terms $\sum_{n=1}^{\infty}\left(\frac{1}{5n}\right)$ diverges, therefore by the comparison test (2.12) part (II) we can conclude the given series with larger terms $\sum_{n=1}^{\infty}\left(\frac{1}{5n-1}\right)$ diverges.

In Example 20 applying the comparison test (2.12) part (II) says that if the series with the smaller terms $a_n = \frac{1}{5n}$ diverges then the series with the larger terms $b_n = \frac{1}{5n-1}$ diverges.
In Example 21 below we have omitted most of the explanation, see if you can follow the mathematics especially the notation.

Example 21

Show that $\sum_{n=1}^{\infty}\left(\tan\left(\frac{\pi}{4n}\right)\right)$ diverges.

[Hint: $\forall x \in \left]0, \frac{\pi}{2}\right[, \quad \tan(x) > x$]

Solution.

By hint with $x = \frac{\pi}{4n}$ we have

$$\tan\left(\frac{\pi}{4n}\right) > \frac{\pi}{4n} = \pi\left(\frac{1}{4n}\right)$$

The series $\sum_{n=1}^{\infty}\left(\frac{1}{4n}\right)$ diverges by Example 19 with $k=4$ therefore $\sum_{n=1}^{\infty}\pi\left(\frac{1}{4n}\right)$ diverges. Hence by the comparison test (2.12) part (II) the given series $\sum_{n=1}^{\infty}\left(\tan\left(\frac{\pi}{4n}\right)\right)$ diverges.

Again we repeat that to apply the comparison test you need a stock of series that you know converge or diverge. Perhaps the simplest series to compare with is the p-series or the geometric series. *Why?*
Because you can identify whether these series converge or diverge by inspection.

D3 **Limit Comparison Test**

Limit Comparison Test (2.13)

Let (a_n) and (b_n) be real positive sequences and suppose the following limit exists:

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L \quad \text{where } L \in \mathbb{R}$$

(A) If $L \neq 0$ [Not Zero] then $\sum(a_n)$ is convergent $\Leftrightarrow$ $\sum(b_n)$ is convergent.

(B) If $L \neq 0$ [Not Zero] then $\sum(a_n)$ is divergent $\Leftrightarrow$ $\sum(b_n)$ is divergent.

Proof. See Exercise 2d

(2.12) (II) If $0 \le a_n \le b_n$ $\quad \sum(a_n)$ is divergent implies $\sum(b_n)$ is divergent

What does the Limit Comparison test state?
The Limit Comparison test (2.13) says that if the limit exists and is equal to

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)=L\neq 0 \quad \text{[Not Zero]}$$

Then part (A) means that if $\sum(a_n)$ is convergent then $\sum(b_n)$ is convergent and also if $\sum(b_n)$ is convergent then $\sum(a_n)$ is convergent.
Remember the mathematical logic symbol $\Leftrightarrow$ means the implication sign goes both ways. That is let P and Q be statements then

$$P\Leftrightarrow Q$$

means that 'if P then Q' and also 'if Q then P'.
Therefore (2.13) part (B) means that if $\sum(a_n)$ is divergent then $\sum(b_n)$ is divergent and also if $\sum(b_n)$ is divergent then $\sum(a_n)$ is divergent.

Example 22

Test $\sum_{n=2}^{\infty}\left(\frac{1}{n^2-n}\right)$ for convergence.

Solution

Can we use the normal comparison test (2.12) for the given series?
Yes we can but we need to find a series that we can compare with and the correct inequality. This will take time and effort to establish because if we try to compare with our normal p-series term $\frac{1}{n^2}$ then we have the following:

$$n^2-n<n^2$$

$$\frac{1}{n^2-n}>\frac{1}{n^2}$$

And we know by the p-series that $\sum\left(\frac{1}{n^2}\right)$ converges. But our inequality sign > is going the wrong way to use normal comparison test (2.12). Therefore to use the normal comparison test we will have to find another series that we can compare with. *But which one, there are lots of them?* Clearly this will take time to find. Another option is to play around with the inequality.
Alternatively we can apply the limit comparison test (2.13).

Let $a_n=\frac{1}{n^2}$ and $b_n=\frac{1}{n^2-n}$ then

$$L = \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{1}{n^2} \div \frac{1}{n^2 - n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n^2} \times \frac{n^2 - n}{1}\right) \quad \left[\begin{array}{l}\text{Inverting the Second Fraction}\\ \text{and Multiplying}\end{array}\right]$$

$$= \lim_{n\to\infty}\left(\frac{n^2 - n}{n^2}\right) \quad \left[\begin{array}{l}\text{Multiplying Numerator}\\ \text{and Denominator}\end{array}\right]$$

$$= \lim_{n\to\infty}\left(1 - \frac{1}{n}\right) \quad \left[\text{Because } \frac{n^2 - n}{n^2} = \frac{n^2}{n^2} - \frac{n}{n^2} = 1 - \frac{1}{n}\right]$$

$$L = 1 - 0 = 1 \neq 0 \quad [\text{Not Zero}]$$

Since $\sum(a_n) = \sum\left(\frac{1}{n^2}\right)$ converges and $L = 1$ therefore by

(2.13) (A) If $L \neq 0$ then $\sum(a_n)$ is convergent $\Leftrightarrow$ $\sum(b_n)$ is convergent

the given series $\sum(b_n) = \sum_{n=2}^{\infty}\left(\frac{1}{n^2 - n}\right)$ converges.

Clearly the Limit Comparison test is easier to use then the normal comparison test in the above example. Since the Limit Comparison test has the symbol $\Leftrightarrow$ therefore we can nominate a_n and b_n the other way round, that is $a_n = \frac{1}{n^2 - n}$ and $b_n = \frac{1}{n^2}$ and then determine the limit. Try it!

Example 23

Test $\sum_{n=1}^{\infty}\left(\frac{1}{3^n - 1}\right)$ for convergence.

Solution.

Can we apply the normal comparison test (2.12) for the given series?

Again it will take time and effort to use the normal comparison test because if we try comparing with $\sum\left(\frac{1}{3^n}\right)$ then examining the denominators first gives:

$$3^n - 1 < 3^n$$

$$\frac{1}{3^n - 1} > \frac{1}{3^n} \quad [\text{By } (*)]$$

We know the series $\sum\left(\frac{1}{3^n}\right)$ converges because it is a geometric series with a common ratio $r = \frac{1}{3} < 1$. But our inequality sign in the above is going in the wrong direction > to use (2.12). Let's try the limit comparison test (2.13):

Let $a_n = \frac{1}{3^n}$ and $b_n = \frac{1}{3^n - 1}$ then

(*) $x \le y$ implies $\frac{1}{x} \ge \frac{1}{y}$

$$L=\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)=\lim_{n\to\infty}\left(\frac{1}{3^n}\div\frac{1}{3^n-1}\right)$$

$$=\lim_{n\to\infty}\left(\frac{1}{3^n}\times\frac{3^n-1}{1}\right) \qquad \left[\begin{array}{l}\text{Inverting the Second Fraction}\\ \text{and Multiplying}\end{array}\right]$$

$$=\lim_{n\to\infty}\left(\frac{3^n-1}{3^n}\right) \qquad [\text{Simplifying}]$$

$$=\lim_{n\to\infty}\left(1-\frac{1}{3^n}\right) \qquad \left[\text{Because } \frac{3^n-1}{3^n}=\frac{3^n}{3^n}-\frac{1}{3^n}=1-\frac{1}{3^n}\right]$$

$$L=1-0=1\neq 0 \qquad [\text{Not Zero}]$$

Since $\sum(a_n)=\sum\left(\frac{1}{3^n}\right)$ converges and $L=1$ therefore by

(2.13) (A) If $L\neq 0$ then $\sum(a_n)$ is convergent $\Leftrightarrow$ $\sum(b_n)$ is convergent

the given series $\sum(b_n)=\sum_{n=1}^{\infty}\left(\frac{1}{3^n-1}\right)$ converges.

Example 24

Test $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n+1}}\right)$ for convergence.

Solution

We can try applying the normal comparison test (2.12) by comparing the given infinite series with $\sum\left(\frac{1}{\sqrt{n}}\right)$. The denominator of general term of this series is $\sqrt{n}$ and we have

$$\sqrt{n+1}>\sqrt{n}$$

$$\frac{1}{\sqrt{n+1}}<\frac{1}{\sqrt{n}}$$

However we know $\sum\left(\frac{1}{\sqrt{n}}\right)$ diverges because $\frac{1}{\sqrt{n}}=\frac{1}{n^{1/2}}$ and so $p=\frac{1}{2}<1$.

Again the inequality is in the wrong direction to use the normal comparison test (2.12). We can try the limit comparison test (2.13):

Let $a_n=\frac{1}{\sqrt{n}}$ and $b_n=\frac{1}{\sqrt{n+1}}$ then

$$L = \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{1}{\sqrt{n}} \div \frac{1}{\sqrt{n+1}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{\sqrt{n}} \times \frac{\sqrt{n+1}}{1}\right)$$

$$= \lim_{n\to\infty}\left(\frac{\sqrt{n+1}}{\sqrt{n}}\right) \qquad \text{[Simplifying]}$$

$$= \lim_{n\to\infty}\left(\sqrt{\frac{n+1}{n}}\right) \qquad \left[\text{Because } \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}\right]$$

$$= \lim_{n\to\infty}\left(\sqrt{1+\frac{1}{n}}\right) = \sqrt{1+0} = 1 \neq 0 \quad \text{[Not Zero]}$$

Since $\sum(a_n) = \sum\left(\frac{1}{\sqrt{n}}\right)$ diverges and $L = 1$ therefore by

(2.13) (B) If $L \neq 0$ then $\sum(a_n)$ is divergent $\Leftrightarrow$ $\sum(b_n)$ is divergent.

the given series $\sum(b_n) = \sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n+1}}\right)$ diverges.

Note that to apply the comparison tests we need a bank of series that we know converge and diverge. In both comparison tests we compare the given series in the question with the series that we know converges or diverges.

The difficulty with applying the comparison tests is that we should have a "feeling" from the outset whether the given series converges or diverges. If we think the given series converges then we know that we need to compare with a converging series which term by term is larger. If our initial hunch is wrong then we will end up putting a lot of effort to examine for divergence when the actual series converges. This is not a waste of time because this is a good way of learning mathematics. The most useful series to compare with is the p-series or the geometric series. Versions of the harmonic series are also helpful to compare with.

Exercise 2d will take a long time to complete because when you first attempt problems on the topic of comparison tests it takes time to digest.

SUMMARY

The p-series (2.11). The series

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^p}\right) \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}$$

Comparison Test (2.12).

If $\forall n \in \mathbb{N}$ (that is for all natural numbers), $0 \leq a_n \leq b_n$ then

(I) $\sum(b_n)$ is convergent $\Rightarrow$ $\sum(a_n)$ is convergent

(II) $\sum(a_n)$ is divergent $\Rightarrow$ $\sum(b_n)$ is divergent

Part (I) says that if the series containing the larger terms (b_n) converges then the series with the smaller terms (a_n) converges.

Part (II) says that if the series with the smaller terms (a_n) diverges then the series with the larger terms (b_n) diverges.

Limit Comparison Test (2.13)

Let (a_n) and (b_n) be real positive sequences and suppose the following limit exists:

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)=L \quad \text{where } L\in\mathbb{R}$$

(A) If $L\neq 0$ [Not Zero] then $\sum(a_n)$ is convergent $\Leftrightarrow$ $\sum(b_n)$ is convergent.

(B) If $L\neq 0$ [Not Zero] then $\sum(a_n)$ is divergent $\Leftrightarrow$ $\sum(b_n)$ is divergent.