Chapter 2 : Infinite Series

Section F **Integral Test**

By the end of this section you will be able to

- evaluate improper integrals
- test a series for convergence by applying the integral test
- apply the integral test to prove the p-series test

F1 **Improper Integrals**

To apply the integral test you need to be familiar with integration in particular improper integrals. We will be using improper integrals throughout this section. *What does the term improper integral mean?*

An integral in which one or both of the limits of integration is infinite. An example is

$$\int_1^\infty f(x)\,dx$$

which is normally evaluated by

$$\lim_{M\to\infty}\left(\int_1^M f(x)\,dx\right) \quad \text{where } M>0$$

If this is a finite value then the limit exists and we say the integral is **convergent**. This is the same term as we use in this chapter for infinite series. *If the limit does* ***not*** *exist then what can we say about the above integral?*

The integral is **divergent** that is if the improper integral does **not** have a finite value then it diverges.

Definition (2.15). Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined for $x\ge a$ where a is a real number. Then the improper integral is defined by

$$\int_a^\infty f(x)\,dx = \lim_{M\to\infty}\left(\int_a^M f(x)\,dx\right) \quad \text{where } M>0$$

If this integral has a finite value then the improper integral exists and we say the integral converges. If the limit does **not** exist then the improper integral diverges.

There is another type of improper integral that is when the integrand is infinite within the region of integration but we will only concentrate on the improper integral defined in (2.15).

Let's do an example.

Example 30

Determine whether the following improper integral

$$\int_0^\infty \left(e^{-x}\right)dx$$

converges. If it does converge determine its value.

Solution.

Let $M\in\mathbb{R}$ and M be positive then by definition (2.15) above we have

$$\int_0^\infty \left(e^{-x}\right)dx = \lim_{M\to\infty}\left(\int_0^M \left(e^{-x}\right)dx\right) \qquad (*)$$

Let's examine the definite integral in (*)

$$\int_0^M (e^{-x})\,dx = \left[-e^{-x}\right]_0^M \qquad \left(\text{Because } \int (e^{-x})\,dx = -e^{-x}\right)$$

$$= -\left[e^{-M} - \underbrace{e^0}_{=1}\right] \qquad \left(\begin{array}{l}\text{Substituting Limits and} \\ \text{Taking out the Negative Sign}\end{array}\right)$$

$$= -\left[e^{-M} - 1\right] = 1 - e^{-M} \qquad (\text{Taking in the Negative Sign})$$

Substituting this, $\int_0^M (e^{-x})\,dx = 1 - e^{-M}$, into (*) gives

$$\int_0^\infty (e^{-x})\,dx = \lim_{M\to\infty}\left(1 - e^{-M}\right)$$

$$= 1 - \underbrace{\lim_{M\to\infty}\left(e^{-M}\right)}_{=0} = 1 - 0 = 1$$

Why is $\lim_{M\to\infty}\left(e^{-M}\right) = 0$ *?*

Because $\lim_{M\to\infty}\left(e^{-M}\right) = \lim_{M\to\infty}\left(\frac{1}{e^M}\right) = 0$, since $e^M \to \infty$ as $M \to \infty$ therefore $\frac{1}{e^M} \to 0$.

Hence the given improper integral $\int_0^\infty (e^{-x})\,dx$ converges with a value of 1.

Example 31

Determine whether the following improper integral converges

$$\int_1^\infty \left(\frac{dx}{x}\right)$$

If it does converge determine its value.

Solution.

Let $M \in \mathbb{R}$ and M be positive then by definition (2.15) we have

$$\int_1^\infty \left(\frac{dx}{x}\right) = \lim_{M\to\infty}\left(\int_1^M \left(\frac{dx}{x}\right)\right) \qquad (\dagger)$$

Let's examine the integral $\int_1^M \left(\frac{dx}{x}\right)$ in (†):

$$\int_1^M \left(\frac{dx}{x}\right) = \left[\ln(x)\right]_1^M \qquad \left(\text{Because } \int \left(\frac{dx}{x}\right) = \ln(x)\right)$$

$$= \left[\ln(M) - \underbrace{\ln(1)}_{=0}\right] = \ln(M)$$

Substituting this result $\int_1^M \left(\frac{dx}{x}\right) = \ln(M)$ into (†) gives

$$\int_1^\infty \left(\frac{dx}{x}\right) = \lim_{M\to\infty}\left(\ln(M)\right) = +\infty$$

Hence the given improper integral, $\int_1^\infty \left(\frac{dx}{x}\right)$, diverges.

Why is the topic improper integrals in this chapter on infinite series?
The answer to this question is given in the next section.

F2 **Harmonic Series and the Integral Test**

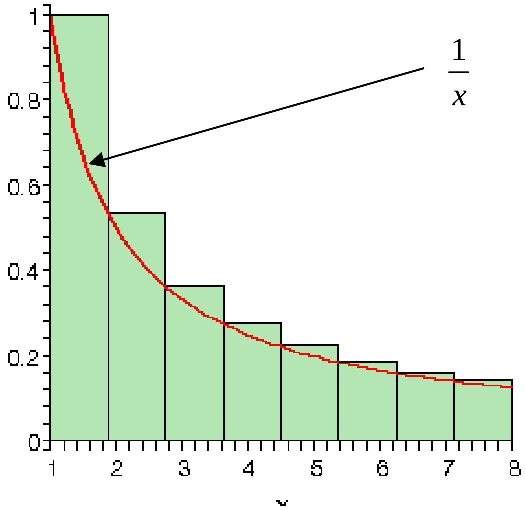
What is the area under the curve $\frac{1}{x}$ *from 1 to* $+\infty$ *equal to?*

This is the integral given by

$$\int_1^\infty \left(\frac{dx}{x}\right)$$

Remember the (definite) integral gives the area under the curve. We can also approximate the area under the curve $f(x)=\frac{1}{x}$ by chopping it up into blocks as shown in Fig 2 below:

Fig 2



We have split the area into rectangular blocks of width 1. *What is the total area of all the blocks?*

$$\text{Area} \simeq (\text{Area of First Block})+(\text{Area of Second Block})+(\text{Area of Third Block})+\ldots$$

What is the area of each block?
Since each is a rectangular block therefore

$$\text{Area of First Block} = 1$$

$$\text{Area of Second Block} = \frac{1}{2}(1)$$

$$\text{Area of Third Block} = \frac{1}{3}(1)$$

$$\begin{aligned}\text{Area} &\simeq 1+\frac{1}{2}(1)+\frac{1}{3}(1)+\frac{1}{4}(1)+\frac{1}{5}(1)+\ldots\\ &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots\\ &= \sum_{n=1}^{\infty}\left(\frac{1}{n}\right)\end{aligned}$$

Do you recognise the infinite series $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)$ *?*

It is the harmonic series discussed earlier in this chapter. The exact area under the curve $\frac{1}{x}$ is given by $\int_1^{\infty}\left(\frac{dx}{x}\right)$ where from above we have $\int_1^{\infty}\left(\frac{dx}{x}\right)=+\infty$ that is it diverges. *Does the series* $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)$ *converge or diverge?*

By examining Fig 2 we can see the areas of blocks is greater than the area under the curve $\frac{1}{x}$ given by the integral $\int_1^{\infty}\left(\frac{dx}{x}\right)=+\infty$ therefore the series $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)=+\infty$. Of course we have already proven that $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)=+\infty$ because it is the well established harmonic series which diverges. Hence the improper integral and the infinite series have the same convergence. This is **not** just the case for $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)$ but is generally the case as given by the following proposition.

Integral Test (2.16). Let $f(x)$ be a function which is

(1) Positive
(2) Continuous
(3) Decreasing or Constant

for $x \geq N$ (generally $N=1$) and is such that

$$f(n)=a_n$$

then

$$\sum_{n=1}^{\infty}(a_n) \text{ converges } \Leftrightarrow \int_1^{\infty}\left(f(x)\,dx\right) \text{ converges}$$

Note: The integral test does **not** tell us what the sum is equal to. *What does the integral test say?*

It says that if a function f satisfies the above 3 conditions for $x \geq N$ $(N=1)$ and

$$f(n)=a_n$$

which means

$$f(1)=a_1,\ f(2)=a_2,\ f(3)=a_3,\ \ldots$$

then $\sum_{n=1}^{\infty}(a_n)$ converges if and only if $\int_1^{\infty}\left(f(x)\,dx\right)$ converges. This sentence says that if we can show $\int_1^{\infty}\left(f(x)\,dx\right)$ has a finite value then $\sum(a_n)$ converges. Hence we only need to check the improper integral for convergence. *Does the integral test imply anything else?*

Yes it is also saying that if $\sum(a_n)$ converges then $\int_1^{\infty}\left(f(x)\,dx\right)$ converges. Remember the mathematical logic symbol $\Leftrightarrow$ means the implication goes both ways.

The contrapositive of the statement is that

$$\int_1^{\infty}\left(f(x)\,dx\right) \text{ diverges } \Leftrightarrow \sum_{n=1}^{\infty}(a_n) \text{ diverges}$$

Hence if the improper integral diverges then the infinite series diverges.
As in other tests the lower limit may **not** start at $n=1$ that is for $n=K$ we have

$$\int_K^{\infty}\left(f(x)\,dx\right) \text{ converges } \Leftrightarrow \sum_{n=K}^{\infty}(a_n) \text{ converges}$$

Proof. Omitted. We will **not prove** the integral test because we need to use some properties of Riemann integration which is beyond our scope.

F3 **Applications of the Integral Test**

Example 32
Test the following series for convergence

$$\sum\left(e^{-2n}\right)$$

Solution
What function, f *, do we use for the integral test in this case?*
Since the general term of the given series is e^{-2n} therefore we examine the function

$$f(x)=e^{-2x}=\frac{1}{e^{2x}} \text{ where } x\geq 1$$

Can we actually apply the integral test?
Need to check the 3 given conditions of the integral test (2.16). Since $f(x)=\dfrac{1}{e^{2x}}$ therefore f is positive and continuous. Also as x increases the denominator, e^{2x}, increases and so $f(x)=\dfrac{1}{e^{2x}}$ decreases. Hence all 3 conditions of the integral test are satisfied. *How do we use the integral test?*
Check the convergence of the analogous improper integral $\int_1^{\infty}\left(e^{-2x}\right)dx$. *But how do we check this?*
By evaluating the improper integral $\int_1^{\infty}\left(e^{-2x}\right)dx$ using definition (2.15)

$$\int_1^{\infty}\left(e^{-2x}\right)dx=\lim_{M\to\infty}\left(\int_1^{M}\left(e^{-2x}\right)dx\right) \qquad (*)$$

where M is a real positive number. Evaluating the integral on the Right Hand Side of (*):

$$\int_1^{M}\left(e^{-2x}\right)dx=\left[\frac{e^{-2x}}{-2}\right]_1^{M} \qquad \left(\text{Because } \int\left(e^{kx}\right)dx=\frac{e^{kx}}{k}\right)$$

$$=-\frac{1}{2}\left[e^{-2M}-e^{-(2\times 1)}\right] \qquad \left(\text{Substituting Limits and Taking Out } -\frac{1}{2}\right)$$

$$=\frac{1}{2}\left[e^{-2}-e^{-2M}\right] \qquad \left(\text{Taking in the Negative Sign}\right)$$

Substituting this $\int_1^M \left(e^{-2x}\right)dx = \frac{1}{2}\left[e^{-2} - e^{-2M}\right]$ into (*) gives

$$\int_1^\infty \left(e^{-2x}\right)dx = \lim_{M\to\infty}\left(\frac{1}{2}\left[e^{-2} - e^{-2M}\right]\right)$$

$$= \frac{1}{2}\left[e^{-2} - \lim_{M\to\infty}\left(e^{-2M}\right)\right]$$

$$= \frac{1}{2}e^{-2} \qquad \left(\text{Because } \lim_{M\to\infty}\left(e^{-2M}\right) = \lim_{M\to\infty}\left(\frac{1}{e^{2M}}\right) = 0\right)$$

The improper integral, $\int_1^\infty \left(e^{-2x}\right)dx$, has a finite value, $\frac{1}{2}e^{-2}$, therefore it converges. By the integral test

(2.16) $$\int_1^\infty \left(f(x)dx\right) \text{ converges } \Leftrightarrow \sum_{n=1}^\infty \left(a_n\right) \text{ converges}$$

the given series $\sum\left(e^{-2n}\right)$ converges.

Note that $\frac{1}{2}e^{-2}$ is **not** the sum of the given infinite series $\sum\left(e^{-2n}\right)$. The improper integral does **not** give the sum of the analogous series but it has the same convergence as the series.

Example 33

Discuss the convergence or divergence of

$$\sum_{n=2}^\infty \left(\frac{1}{n\ln(n)}\right)$$

Solution.

We use the integral test for testing the convergence of the given series.

Which function do we use in the improper integral?

Since the general term of the series is $\frac{1}{n\ln(n)}$ therefore we consider the function

$$f(x) = \frac{1}{x\ln(x)}$$

for $x \geq 2$. *Are the 3 conditions of the integral test satisfied?*

Yes because $f(x)$ is continuous and positive for $x \geq 2$. Since $x\ln(x)$ increases as x increases therefore $\frac{1}{x\ln(x)}$ decreases so it is a decreasing function. Hence we can use the integral test. *How?*

We test the convergence of the analogous improper integral by evaluating $\int_2^\infty \left(\frac{dx}{x\ln(x)}\right)$. *How?*

Using definition (2.15)

$$\int_2^\infty \left(\frac{dx}{x\ln(x)}\right) = \lim_{M\to\infty}\left(\int_2^M \left(\frac{dx}{x\ln(x)}\right)\right) \qquad (\dagger)$$

How do we determine the integral on the Right Hand Side of (†)?

By using substitution, let

$$u = \ln(x)$$

$$\frac{du}{dx} = \frac{1}{x} \qquad \left[\text{Differentiating}\right] \qquad dx = (x)du$$

Putting these into the above integral gives

$$\int\left(\frac{dx}{x\ln(x)}\right) = \int\left(\frac{\cancel{x}du}{\cancel{x}u}\right) \qquad \left[\text{Substituting } \ln(x) = u \text{ and } dx = xdu\right]$$

$$= \int\left(\frac{du}{u}\right) = \ln(u) = \ln\left[\ln(x)\right] \qquad \left[\text{Substituting } u = \ln(x)\right]$$

Evaluating the definite integral

$$\int_2^M\left(\frac{dx}{x\ln(x)}\right) = \left[\ln(\ln(x))\right]_2^M \qquad \left[\text{Substituting } \int\frac{dx}{x\ln(x)} = \ln\left[\ln(x)\right]\right]$$

$$= \ln(\ln(M)) - \ln(\ln(2)) \qquad \left[\text{Substituting the Limits}\right]$$

Putting this result $\int_2^M\left(\frac{dx}{x\ln(x)}\right) = \ln(\ln(M)) - \ln(\ln(2))$ into (†) gives

$$\int_2^\infty\left(\frac{dx}{x\ln(x)}\right) = \lim_{M\to\infty}\left[\ln(\ln(M)) - \ln(\ln(2))\right] = +\infty$$

Because $\ln(\ln(M)) \to \infty$ as $M \to \infty$.

Hence the improper integral, $\int_2^\infty\left(\frac{dx}{x\ln(x)}\right)$, does not have a finite value so it diverges.

By the contrapositive of the integral test

(2.16) $$\int\left(f(x)dx\right) \text{ diverges } \Leftrightarrow \sum(a_n) \text{ diverges}$$

the given series, $\sum_{n=2}^{\infty}\left(\frac{1}{n\ln(n)}\right)$, diverges.

The integral test says that if the analogous improper integral diverges then the series diverges. Also if the improper integral converges then the series converges.

Example 34

Discuss the convergence or divergence of

$$\sum\left(\frac{e^{\tan^{-1}(n)}}{n^2+1}\right)$$

Solution.

Can we apply the integral test for the convergence of the given series?

Let's try. *Which function do we consider for the improper integral?*

Since the general term of the series is $\frac{e^{\tan^{-1}(n)}}{n^2+1}$ we examine the function

$$f(x) = \frac{e^{\tan^{-1}(x)}}{x^2+1} \qquad (x \geq 1)$$

Then f is decreasing, positive and continuous for $x \geq 1$, therefore all three conditions of the integral test are satisfied. Consider the analogous improper integral with $M > 0$

$$\int_1^\infty \left(\frac{e^{\tan^{-1}(x)}}{x^2+1} \right) dx = \lim_{M \to \infty} \left(\int_1^M \left(\frac{e^{\tan^{-1}(x)}}{x^2+1} \right) dx \right) \qquad (\dagger\dagger)$$

How do we find the integral on the Right Hand Side of (††), $\int \left(\frac{e^{\tan^{-1}(x)}}{x^2+1} \right) dx$ *?*

Use substitution with

$$u = \tan^{-1}(x)$$

$$\frac{du}{dx} = \frac{1}{1+x^2} \quad [\text{Differentiating}] \qquad dx = (1+x^2)\,du$$

Putting this into the Right Hand Side integral of (††) without limits for the time being:

$$\int \left(\frac{e^{\tan^{-1}(x)}}{x^2+1} \right) dx = \int \left(\frac{e^u}{x^2+1} \right) (x^2+1)\,du \qquad \left[\begin{array}{l} \text{Substituting } \tan^{-1}(x) = u \\ \text{and } \ dx = (1+x^2)\,du \end{array} \right]$$

$$= \int (e^u\,du) \qquad [\text{Cancelling } x^2+1]$$

$$= e^u = e^{\tan^{-1}(x)} \qquad [\text{Substituting } u = \tan^{-1}(x)]$$

Substituting this result $\int \left(\frac{e^{\tan^{-1}(x)}}{x^2+1} \right) dx = e^{\tan^{-1}(x)}$ into the Right Hand Side of (††) we have

$$\lim_{M \to \infty} \left(\int_1^M \left(\frac{e^{\tan^{-1}(x)}}{x^2+1} \right) dx \right) = \lim_{M \to \infty} \left[e^{\tan^{-1}(x)} \right]_1^M$$

$$= \lim_{M \to \infty} \left[e^{\tan^{-1}(M)} - e^{\tan^{-1}(1)} \right] \qquad (\text{Substituting Limits})$$

$$= \left(e^{\lim_{M \to \infty} \left[\tan^{-1}(M) \right]} \right) - e^{\tan^{-1}(1)}$$

$$= e^{\frac{\pi}{2}} - e^{\frac{\pi}{4}} \qquad \left(\begin{array}{l} \text{Because } \lim_{M \to \infty} \left[\tan^{-1}(M) \right] = \frac{\pi}{2} \\ \text{and } \tan^{-1}(1) = \frac{\pi}{4} \end{array} \right)$$

The improper integral, $\int_1^\infty \left(\frac{e^{\tan^{-1}(x)}}{x^2+1} \right) dx$, has a finite value, $e^{\pi/2} - e^{\pi/4}$, so it converges.

Therefore by the integral test

(2.16) $$\int_1^\infty (f(x)\,dx) \text{ converges } \Leftrightarrow \sum_{n=1}^\infty (a_n) \text{ converges}$$

the given series, $\sum \left(\frac{e^{\tan^{-1}(n)}}{n^2+1} \right)$, converges.

Example 35

Prove the p-series test (2.11) of Section D.

The p-series is

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^p}\right)\begin{cases}\text{converges if } p>1\\ \text{diverges if } p\le 1\end{cases}$$

Solution.

Which function do we consider?

Let $f:\left]1,\ \infty\right]\to\mathbb{R}^+$ be given by

$$f(x)=\frac{1}{x^p}\qquad \text{where } p\ge 0$$

Then f is decreasing (or constant), positive and continuous for $x\ge 1$ and $p\ge 0$. Hence f satisfies the conditions for using the integral test. Consider the analogous improper integral

$$\int_1^{\infty}\left(\frac{dx}{x^p}\right)=\lim_{M\to\infty}\left(\int_1^{M}\left(\frac{dx}{x^p}\right)\right)\qquad (\dagger\dagger\dagger)$$

Examine the case for p not equal to 1, $p\ne 1$:

What is the integral on the Right Hand Side in (†††) equal to?

$$\int_1^{M}\left(\frac{dx}{x^p}\right)=\int_1^{M}\left(x^{-p}\right)dx\qquad\left(\text{Rewriting } \frac{1}{x^p}=x^{-p}\right)$$

$$=\left[\frac{x^{-p+1}}{-p+1}\right]_1^M\qquad(\text{Integrating})$$

$$=\frac{M^{1-p}-1}{1-p}\qquad(\text{Substituting the limits})$$

Case 1. Let $p>1$. Substituting the above result $\int_1^{M}\left(\frac{dx}{x^p}\right)=\frac{M^{1-p}-1}{1-p}$ into (†††) gives

$$\int_1^{\infty}\left(\frac{dx}{x^p}\right)=\lim_{M\to\infty}\left(\frac{M^{1-p}-1}{1-p}\right)$$

$$=\frac{\left(\lim_{M\to\infty}\left(M^{1-p}\right)-1\right)}{1-p}=\frac{0-1}{1-p}\qquad\left[\begin{array}{l}\text{Since } p>1 \text{ therefore}\\ \lim_{M\to\infty}\left(M^{1-p}\right)=\lim_{M\to\infty}\left(\frac{1}{M^{p-1}}\right)=0\end{array}\right]$$

$$=\frac{-1}{1-p}=\frac{1}{p-1}\qquad\left[\begin{array}{l}\text{Multiplying Numerator and}\\ \text{Denominator by } -1\end{array}\right]$$

Since the improper integral has a finite value $\frac{1}{p-1}$ therefore it converges for $p>1$.

By the integral test

(2.16) $$\int_1^{\infty}\left(f(x)\,dx\right)\text{ converges}\iff\sum_{n=1}^{\infty}\left(a_n\right)\text{ converges}$$

the given series, $\sum\left(\frac{1}{n^p}\right)$, converges for $p>1$.

What other cases do we need to consider?

Case 2. Let $0\le p<1$. For these values of p the improper integral diverges because

$$\int_1^{\infty}\left(\frac{dx}{x^p}\right)=\lim_{M\to\infty}\left(\frac{M^{1-p}-1}{1-p}\right) \quad \left[\text{Because from above } \int_1^{M}\left(\frac{dx}{x^p}\right)=\frac{M^{1-p}-1}{1-p}\right]$$

$$=+\infty \quad \left[\text{Because } M^{1-p}\to+\infty \text{ as } M\to+\infty \text{ for } 0\le p<1\right]$$

The improper integral diverges for $0\le p<1$ therefore by the integral test

(2.16) $\int\left(f(x)dx\right)$ diverges $\Leftrightarrow$ $\sum(a_n)$ diverges

the given series, $\sum\left(\frac{1}{n^p}\right)$, diverges for $0\le p<1$.

Case 3. Let $p=1$. For this value of p we have the following improper integral:

$$\int_1^{\infty}\left(\frac{dx}{x}\right)=\lim_{M\to\infty}\left(\int_1^{M}\left(\frac{dx}{x}\right)\right)$$

$$=\lim_{M\to\infty}\left[\ln(x)\right]_1^M \qquad \left(\text{Because } \int\left(\frac{dx}{x}\right)=\ln(x)\right)$$

$$=\lim_{M\to\infty}\left[\ln(M)-\underbrace{\ln(1)}_{=0}\right] \qquad (\text{Substituting Limits})$$

$$=\lim_{M\to\infty}\left[\ln(M)\right]=+\infty$$

The improper integral diverges for $p=1$ therefore by the integral test

(2.16) $\int\left(f(x)dx\right)$ diverges $\Leftrightarrow$ $\sum(a_n)$ diverges

the given series, $\sum\left(\frac{1}{n^p}\right)$, diverges for $p=1$.

Case 4. *If $p<0$ we cannot use the integral test, why not?*

Because $\frac{1}{x^p}$ is **not** a decreasing function if $p<0$ so it does not satisfy one of the conditions for the integral test. We need to use another test. *But which one?*

If $p<0$ then the series $\sum\left(\frac{1}{n^p}\right)$ diverges because the nth term does **not** converge to 0. That is $\lim_{n\to\infty}\left(\frac{1}{n^p}\right)\ne 0$ [Not Zero] for $p<0$. So by (2.6) the series $\sum\left(\frac{1}{n^p}\right)$ diverges for $p<0$.

Collecting all the above cases we have proven the p-series test:

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^p}\right)\begin{cases}\text{converges if } p>1\\ \text{diverges if } p\le 1\end{cases}$$

Note that we can ONLY apply the integral test if the 3 conditions – positive, continuous and decreasing (or constant) are satisfied for the analogous function of the series.

(2.6) If $\lim_{n\to\infty}(a_n)\ne 0$ then $\sum(a_n)$ diverges

Example 36

Discuss the convergence or divergence of the following series $\sum\left(e^n\right)$.

Solution

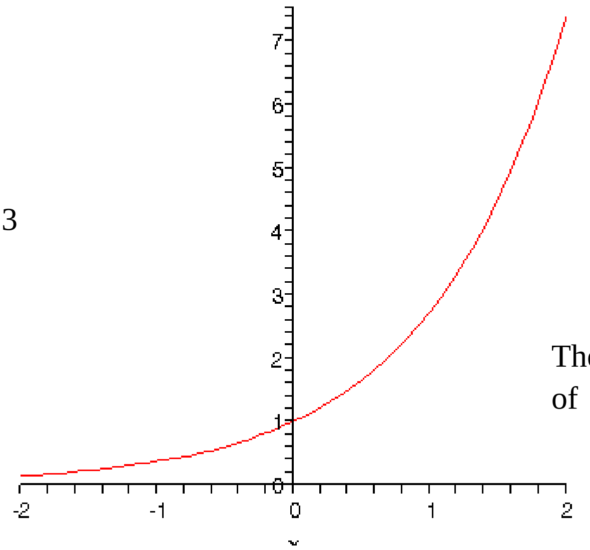
If we want to apply the integral test then we need to consider the analogous function of the given general term e^n. *What function should we consider?*

$$f(x)=e^x \quad (x\geq 1)$$

Can we use the integral test?

No because e^x is an increasing function.

Fig 3



The Graph of e^x

Hence we cannot apply the integral test because it does **not** satisfy one of the conditions of the test. *So how do we test the given series* $\sum\left(e^n\right)$ *?*

Since the nth term $\lim_{n\to\infty}\left(e^n\right)\neq 0$ [Not Zero] therefore by (2.6) the given series diverges.

SUMMARY

Integral Test (2.16). Let $f(x)$ be a function which is

(1) Positive
(2) Continuous
(3) Decreasing or Constant

for $x\geq N$ (generally $N=1$) and is such that

$$f(n)=a_n$$

then

$$\sum_{n=1}^{\infty}\left(a_n\right) \text{ converges} \Leftrightarrow \int_1^{\infty}\left(f(x)\,dx\right) \text{ converges}$$

We initially check the above three conditions and then test the improper integral for convergence because the analogous improper integral and series have the same convergence.

(2.6) If $\lim_{n\to\infty}\left(a_n\right)\neq 0$ then $\sum\left(a_n\right)$ diverges