CHAPTER 1

Logic and Proof

The mathematics we cover in this chapter is called Logic and Proof. At first you will find this a difficult and challenging chapter and **not** the kind of mathematics you would have been familiar with. However to understand this chapter you need to follow each step at a slower pace and also become comfortable with the notation used. We will regularly be using the results of this chapter so it is important that you understand the material thoroughly. Most students struggle with the topics in this chapter because they don't know what is required of them.
Some sections will seem abstract and lacking in applications but the practical use of the theory lies in digital electronics and computer science.
The proof part of this chapter is particularly difficult because each proof uses different concepts and it is not a handle turning exercise.

SECTION A **Propositional Logic**

By the end of this section you will be able to

- Recognise a proposition
- Recognise a predicate
- Negate a proposition
- Write the connectives AND, OR, NOT, IMPLICATION in symbolic form
- Form a compound proposition (or statement) using the symbolic form of NOT, AND, OR, IMPLICATION

A1 **Propositions and Predicates**

What does the term proposition mean in mathematics?
It is a statement which has a value of true or false. *Which of the following are propositions?*

(a) Arsenal Football club won the double in 2002.
(b) 5+3=7
(c) $x+5=2$
(d) Liz Hurley looks beautiful.
(e) The light is off.
(f) The world ended on 6^{th} June 1984.

(a), (b) (e) and (f) are propositions but (c) and (d) are **not** propositions. *Why not?*
(c) is not a proposition because it contains an unknown, x. Since we do not know the value of x therefore we cannot say whether $x+5=2$ is true or false. 'Liz Hurley looks beautiful' is **not** a proposition because beauty is subjective.
Note that a proposition can be **false** such as (b) 5+3=7 and (f) The world ended on 6^{th} June 1984.
The statement $x+5=2$ is an example of a **predicate**. *What does predicate mean?*
It is a statement with a unknown such as x. Once we substitute values for all the unknowns then it becomes a proposition. For example if we substitute $x=3$ then

$$3+5=2$$

is a (false) proposition.
We normally denote statements by the letters P, Q , R, S etc.

Example 1

Decide which of the following statements are propositions:

P : 50% off Recommended Retail Price of £100

Q : Pigs will fly

R : The world is getting warmer

S : A man can be pregnant

Solution

P, Q and S are propositions because they are either true or false. However R is **not** a proposition because 'The world is getting warmer' is subjective.

If a statement is subjective or uses the pronouns he or she then the statement **cannot** be a proposition. For example the statement 'he is a good football player' is not a proposition because it has the pronoun 'he'. However if you changed the statement to 'Henry is a good football player' then this is a proposition.

Also if it has unknown (x) then it is not a proposition.

A2 **Negation**

What does negation mean?

Generally means the opposite of something. If P is a proposition then the negation of P is not P and it is denoted by $\neg P$ or (not P).

Let P : I weigh more than 75kg. *What is $\neg P$ equal to?*

The opposite of weighing more than 75kg is **not** weighing more than 75kg, hence

$\neg P$: I do **not** weigh more than 75kg

This can also be written as

(not P)= I do not weigh more than 75kg

Let Q be the proposition 'I am dressed'

What is $\neg Q$ equal to?

$\neg Q$ means (not Q) or the opposite of Q which is 'I am undressed'.

Example 2

Write the negations of the following propositions:

(1) $a < b$

(2) $2+2 \neq 5$

(3) It is not raining

(4) Bristol is east of Edinburgh

(5) You have not passed the exam

(6) Alan is taller than Don

Solution The negation of each of the above is the following:

(1) $a \geq b$ because the opposite of $a < b$ is a is greater than or equal to b.

(2) $2+2=5$ because the opposite of does not equal $(\neq)$ is equal (=).

(3) It is raining

(4) Bristol is not east of Edinburgh

(5) You have passed the exam

(6) Alan is not taller than Don or you could say Alan is shorter or have the same height as Don.

Note the answer to (1) is not $a > b$ but $a \geq b$ because you have to cover **all** possibilities. That is P and $\neg P$ (not P) has to include all possibilities.
Also notice the answer to (3) that is

Not (not raining) means it is raining

Similarly the opposite to 'not passing an exam' is 'passing an exam'
In general terms not (not P) is P. Hence $\neg(\neg P) = P$
Moreover if P is true then $\neg P$ is false and vice-versa.
The next example is more difficult, see if you can answer it without looking at the solution.

Example 3

Negate the following statement:

P : There are integers a and b such that $\frac{a}{b} = \sqrt{2}$.

Solution

Negate means not P or in symbolic form $\neg P$.

$\neg P$; There are NO integers a and b such that $\frac{a}{b} = \sqrt{2}$.

A3 **AND**

Two propositions can be combined by the word 'and' to form a compound proposition. Let P and Q be propositions then P **and** Q is denoted by

$$P \wedge Q$$

The compound proposition $P \wedge Q$ is called the **conjunction** of the original propositions.

Example 4

Let P : Grass is blue
Q : Pigs will fly

Form a sentence which describes $P \wedge Q$.

Solution

The notation $P \wedge Q$ means p and q.

$P \wedge Q$: $\underbrace{\text{Grass is blue}}_{P}$ $\underbrace{\text{and}}_{\wedge}$ $\underbrace{\text{pigs will fly}}_{Q}$.

Example 5

Let P : Models weigh less than 60kg
Q : Models dress size should not exceed 10

Form a sentence which describes

(i) $P \wedge Q$ (ii) $P \wedge (\neg Q)$ (iii) $(\neg P) \wedge (\neg Q)$

Solution

(i) $P \wedge Q$: $\underbrace{\text{Models weigh less than 60kg}}_{P}$ $\underbrace{\text{and}}_{\wedge}$ $\underbrace{\text{their dress size should not exceed 10}}_{Q}$.

(ii) The notation $P \wedge (\neg Q)$ means P and $(\text{not } Q)$.

$P \wedge (\neg Q)$: $\underbrace{\text{Models weigh less than 60kg}}_{P}$ $\underbrace{\text{and}}_{\wedge}$ $\underbrace{\text{their dress size is greater than 10}}_{\neg Q}$.

(iii) The notation $(\neg P) \wedge (\neg Q)$ means $(\text{not } P)$ and $(\text{not } Q)$.

$(\neg P) \wedge (\neg Q)$: $\underbrace{\text{Models weigh 60kg or more}}_{\neg P}$ $\underbrace{\text{and}}_{\wedge}$ $\underbrace{\text{their dress size is greater than 10}}_{\neg Q}$.

Example 6

Let P : 2 is the only even prime number

Q: 3 is an odd number

Form a sentence which describes

(a) $P \wedge Q$ (b) $(\neg P) \wedge Q$ (c) $P \wedge (\neg Q)$ (d) $(\neg P) \wedge (\neg Q)$

Solution

(a) $P \wedge Q$: $\underbrace{\text{2 is the only even prime number}}_{P}$ $\underbrace{\text{and}}_{\wedge}$ $\underbrace{\text{3 is an odd number}}_{Q}$

(b) $(\neg P) \wedge Q$: $\underbrace{\text{2 is NOT the only even prime number}}_{\neg P}$ $\underbrace{\text{and}}_{\wedge}$ $\underbrace{\text{3 is an odd number}}_{Q}$

(c) $P \wedge (\neg Q)$: $\underbrace{\text{2 is the only even prime number}}_{P}$ $\underbrace{\text{and}}_{\wedge}$ $\underbrace{\text{3 is NOT an odd number}}_{\neg Q}$

(d)

$(\neg P) \wedge (\neg Q)$: $\underbrace{\text{2 is NOT the only even prime number}}_{\neg P}$ $\underbrace{\text{and}}_{\wedge}$ $\underbrace{\text{3 is NOT an odd number}}_{\neg Q}$

A4 **OR**

Two propositions can also be combined by the word 'or'. Let P and Q be propositions then P **o**r Q is denoted by

$$P \vee Q$$

A sentence of the form 'P or Q' is called a **disjunction** and is denoted by $P \vee Q$.

A symbol which combines two or more statements is called a connective. For example $\wedge$ and $\vee$ are connectives.

Example 7

Let P : Mercedes cars have no engine

Q: Mercedes has a symmetrical logo

Write the following in symbolic form:

(a) Mercedes cars have no engine or they have a symmetrical logo.

(b) Mercedes cars do have an engine.

(c) Mercedes does not have a symmetrical logo or an engine.

Solution

(a) This is P or Q, hence in symbolic form we have $P \vee Q$.

(b) $\neg P$ (Not no engine means there is an engine. That is Mercedes does have an engine is not P).

(c) This is (not Q) or P, hence in symbolic form we have $(\neg Q) \vee P$

Example 8

Let $P:\ 3<4$

$Q:\ 4<3$

Write out the following

(a) $P \vee Q$ (b) $\neg P$ (c) $(\neg P)\vee(\neg Q)$

Solution

(a) The notation $P \vee Q$ means P or Q

$$P \vee Q:\ 3<4 \text{ or } 4<3$$

(b) $\neg P:\ 3 \geq 4$ (The opposite of 3<4 is $3 \geq 4$).

(c) $(\neg P)\vee(\neg Q):\ 3 \geq 4$ or $4 \geq 3$

Example 9

Let P : A natural number is prime

Q: A natural number is composite

Write out the following:

(a) $P \vee Q$ (b) $(\neg P)\vee(\neg Q)$

Solution

a) The notation $P \vee Q$ means P or Q.

$P \vee Q$: A natural number is either prime or composite.

b) The notation $(\neg P)\vee(\neg Q)$ means $(\text{not } P)$ or $(\text{not } Q)$.

$(\neg P)\vee(\neg Q)$: A natural number is NOT prime nor composite.

A5 **IMPLICATION**

Let P and Q be two statements. The compound statement

‘P implies Q’

means ‘if P then Q’. Implication is denoted by the symbol $\Rightarrow$. That is

$$P \Rightarrow Q$$

which says P implies Q. For implication P, Q do **not** need to be propositions, we can relax P and Q to be statements.

Example 10

Let P : I am rich

Q: I take my holidays in Mauritius

Form a sentence that describes

(i) $P \Rightarrow Q$ (ii) $Q \Rightarrow P$

Solution

(i) The notation $P \Rightarrow Q$ means ‘if P then Q’

$$\text{If } \underbrace{\text{I am rich}}_{P} \text{ then } \underbrace{\text{I take my holidays in Mauritius}}_{Q}.$$

(ii) The notation $Q \Rightarrow P$ means ‘if Q then P’.

$$\text{If } \underbrace{\text{I take my holidays in Mauritius}}_{Q} \text{ then } \underbrace{\text{I am rich}}_{P}.$$

Example 11

Let P : I work hard

Q : I will not pass my exams

Write out a sentence for

(a) $P \Rightarrow (\neg Q)$ (b) $(\neg P) \Rightarrow Q$

Solution

(a) Note that $\neg Q$ is (not Q) and means that 'I will pass my exams'

$P \Rightarrow (\neg Q)$: If $\underbrace{\text{I work hard}}_{P}$ then $\underbrace{\text{I will pass my exams}}_{\neg Q}$.

(b) $(\neg P) \Rightarrow Q$: If $\underbrace{\text{I do not work hard}}_{\neg P}$ then $\underbrace{\text{I will not pass my exams}}_{Q}$.

Example 12

Let P : You are a pop star

Q : You are a banker

R : You will buy a house in Surrey

Write out the following in symbolic form.

If you are a pop star or a banker then you will buy a house in Surrey.

Solution

Between if and then is P or Q, in symbol form we write this as $P \vee Q$. After then is the r statement. Hence

$$\text{If } \underbrace{\text{you are a pop star or a banker}}_{P \vee Q} \text{ then } \underbrace{\text{you will buy a house in Surrey}}_{R}.$$

In symbolic form we have:

$$(P \vee Q) \Rightarrow R$$

The brackets are present because the sentence says **either** you are a banker or a pop star. In either case you will buy a house in Surrey.

Example 13

Write out the following using if and then:

$$x^2 - 1 = 0 \Rightarrow x^2 = 1$$
$$\Rightarrow x = \sqrt{1}$$
$$\Rightarrow x = \pm 1$$

Solution

Each statement carries on from the previous statement.

$$\text{If } x^2 - 1 = 0 \text{ then } x^2 = 1$$
$$\text{If } x^2 = 1 \text{ then } x = \sqrt{1}$$
$$\text{If } x = \sqrt{1} \text{ then } x = \pm 1$$

The implication connective is very important in mathematics because when proofing results the proof consists of a sequence of true statements generally connected by implication. A proof starts with a statement which we know is true and ends with a statement that we are required to prove. Each true statement follows from the previous true statement generally by implication.

Example 14

Let P: ABC is a right-angled triangle with sides $a, \quad b, \quad c$ where $a \le b < c$

Q : The sides of the triangle ABC satisfy the identity $c^2 = a^2 + b^2$

Write out the following:

(i) $P \Rightarrow Q$ (ii) $Q \Rightarrow P$

Solution

(i) $P \Rightarrow Q$ means if P then Q

$P \Rightarrow Q$: If ABC is a right-angled triangle with sides $a, \quad b, \quad c$ where $a \le b < c$ then it satisfies the identity $c^2 = a^2 + b^2$.

(ii) $Q \Rightarrow P$: If the sides of the triangle ABC satisfy the identity $c^2 = a^2 + b^2$ then ABC is a right-angled triangle.

Example 14 is Pythagoras Theorem.

SUMMARY

A proposition is a statement that is true or false.

Not P $(\neg P)$ is the opposite of P. Not P makes a compound proposition.

Other compound propositions can be made by the connectives ‘and’ , ‘or’ denoted by $\wedge, \vee$ respectively.

Implication of two statements, P and Q, represents ‘if P then Q’ and is denoted by $P \Rightarrow Q$