Section E **Evaluating Limits**

By the end of this section you will be able to

- evaluate limits of a sequence
- establish limits of general sequences
- determine what is meant by a null sequence
- use the propositions and theorems of the last two sections to evaluate limits of sequences

E1 **Evaluating Limits**

In this section you need to know how to apply the propositions of the last two sections to determine limits of sequences. For example we use the following limits:

$$\lim_{n\to\infty}\left(\frac{1}{n}\right)=0 \qquad (*)$$

$$\lim_{n\to\infty}\left(\frac{K}{n}\right)=K\lim_{n\to\infty}\left(\frac{1}{n}\right)=0 \text{ where } K \text{ is a constant} \qquad (**)$$

We also use other general results proved earlier:

Let (x_n) and (y_n) be convergent sequences then

(5.15) Part (iii) $\lim_{n\to\infty}(x_n y_n)=\lim_{n\to\infty}(x_n)\lim_{n\to\infty}(y_n)$

(5.15) Part (iv) $\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right)=\frac{\lim_{n\to\infty}(x_n)}{\lim_{n\to\infty}(y_n)}$ where $y_n\neq 0$ and $\lim_{n\to\infty}(y_n)\neq 0$

Exercise 5(d) Question 3 $\lim_{n\to\infty}(Ax_n+By_n)=A\lim_{n\to\infty}(x_n)+B\lim_{n\to\infty}(y_n)$ where A and B are constants.

We will just apply these without giving references to them in the evaluations of limits of sequences in this section.

Example 15

Determine

$$\lim_{n\to\infty}\left(\frac{n+1}{3n+5}\right)$$

Solution

What does this question mean?

Well it says what is the value of $\frac{n+1}{3n+5}$ as $n\to\infty$. *How can we find this value?*

Generally we identify the **dominant term** and then divide numerator and denominator by this dominant term. *What is the dominant term in* $\frac{n+1}{3n+5}$*?*

The dominant term is clearly n so we divide both numerator and denominator of $\frac{n+1}{3n+5}$ by n. Hence

$$\frac{n+1}{3n+5}=\frac{\frac{n}{n}+\frac{1}{n}}{\frac{3n}{n}+\frac{5}{n}} \qquad \left[\text{Using } \frac{a+b}{c}=\frac{a}{c}+\frac{b}{c} \text{ on Numerator and Denominator}\right]$$

$$=\frac{1+\frac{1}{n}}{3+\frac{5}{n}} \qquad [\text{Cancelling the } n\text{'s}]$$

Remember $\lim_{n\to\infty}(K)=K$ where K is a constant. Therefore $\lim_{n\to\infty}(1)=1$ and $\lim_{n\to\infty}(3)=3$.

Since by (*) and (**) we have $\lim_{n\to\infty}\left(\frac{1}{n}\right)=0$ and $\lim_{n\to\infty}\left(\frac{5}{n}\right)=0$ respectively therefore using these and substituting the above gives

$$\lim_{n\to\infty}\left(\frac{n+1}{3n+5}\right)=\lim_{n\to\infty}\left(\frac{1+\frac{1}{n}}{3+\frac{5}{n}}\right) \qquad [\text{By Above}]$$

$$=\frac{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)}{\lim_{n\to\infty}\left(3+\frac{5}{n}\right)}=\frac{1+\lim_{n\to\infty}\left(\frac{1}{n}\right)}{3+\lim_{n\to\infty}\left(\frac{5}{n}\right)}=\frac{1+0}{3+0}=\frac{1}{3}$$

Hence $\lim_{n\to\infty}\left(\frac{n+1}{3n+5}\right)=\frac{1}{3}$.

In section 5C we proved the generic result:

(5.18) $\lim_{n\to\infty}\left(\frac{K}{n^p}\right)=0$ where p is a positive real number and K is a constant.

We use this result to evaluate limits of sequences.

Example 16

Evaluate

$$\lim_{n\to\infty}\left(\frac{3n^2+4n+1}{2n^2-3n+7}\right)$$

Solution

How do we evaluate this limit?

First we identify the **dominant term** and then divide numerator and denominator by this dominant term. *What is the dominant term in this example?*

n^2. Divide numerator and denominator of $\frac{3n^2+4n+1}{2n^2-3n+7}$ by n^2 so we have

$$\frac{3n^2+4n+1}{2n^2-3n+7}=\frac{\frac{3n^2}{n^2}+\frac{4n}{n^2}+\frac{1}{n^2}}{\frac{2n^2}{n^2}-\frac{3n}{n^2}+\frac{7}{n^2}} \qquad \left[\text{Using } \frac{a+b}{c}=\frac{a}{c}+\frac{b}{c} \text{ on Numerator and Denominator}\right]$$

$$=\frac{3+\frac{4}{n}+\frac{1}{n^2}}{2-\frac{3}{n}+\frac{7}{n^2}} \qquad [\text{Cancelling}]$$

Substituting this and using the above limit results, (5.18) and (**), we have

$$\lim_{n\to\infty}\left(\frac{3n^2+4n+1}{2n^2-3n+7}\right)=\lim_{n\to\infty}\left(\frac{3+\frac{4}{n}+\frac{1}{n^2}}{2-\frac{3}{n}+\frac{7}{n^2}}\right) \qquad \text{[By Above]}$$

$$=\frac{3+\lim_{n\to\infty}\left(\frac{4}{n}\right)+\lim_{n\to\infty}\left(\frac{1}{n^2}\right)}{2-\lim_{n\to\infty}\left(\frac{3}{n}\right)+\lim_{n\to\infty}\left(\frac{7}{n^2}\right)} \qquad \left[\begin{array}{l}\text{The constant limits are}\\ \lim_{n\to\infty}(3)=3 \text{ and } \lim_{n\to\infty}(2)=2\end{array}\right]$$

$$=\frac{3+0+0}{2-0+0}=\frac{3}{2} \qquad \text{[By (5.18) and (**)]}$$

Hence $\lim_{n\to\infty}\left(\frac{3n^2+4n+1}{2n^2-3n+7}\right)=\frac{3}{2}$.

Example 17

Evaluate

$$\lim_{n\to\infty}\left(\frac{3n^2+4n+1}{2n^3-1}\right)$$

Solution

How do we evaluate this limit?

Similarly to the above examples by first finding the dominant term and then using the limits already established. *What is the dominant term in* $\frac{3n^2+4n+1}{2n^3-1}$ *?*

n^3. Dividing numerator and denominator by n^3 we have

$$\frac{3n^2+4n+1}{2n^3-1}=\frac{\frac{3n^2}{n^3}+\frac{4n}{n^3}+\frac{1}{n^3}}{\frac{2n^3}{n^3}-\frac{1}{n^3}}=\frac{\frac{3}{n}+\frac{4}{n^2}+\frac{1}{n^3}}{2-\frac{1}{n^3}} \qquad \text{[Cancelling]}$$

Substituting this and evaluating the limit

$$\lim_{n\to\infty}\left(\frac{3n^2+4n+1}{2n^3-1}\right)=\lim_{n\to\infty}\left(\frac{\frac{3}{n}+\frac{4}{n^2}+\frac{1}{n^3}}{2-\frac{1}{n^3}}\right) \qquad \text{[By Above]}$$

$$=\frac{\lim_{n\to\infty}\left(\frac{3}{n}\right)+\lim_{n\to\infty}\left(\frac{4}{n^2}\right)+\lim_{n\to\infty}\left(\frac{1}{n^3}\right)}{2-\lim_{n\to\infty}\left(\frac{1}{n^3}\right)}$$

$$=\frac{0+0+0}{2-0}=\frac{0}{2}=0 \qquad \text{[By (5.18) and (**)]}$$

(5.18) $\lim_{n\to\infty}\left(K/n^p\right)=0$ where p is a positive real number and K is a constant.

(**) $\lim_{n\to\infty}\left(\frac{K}{n}\right)=K\lim_{n\to\infty}\left(\frac{1}{n}\right)=0$ where K is a constant

Hence $\lim_{n\to\infty}\left(\frac{3n^2+4n+1}{2n^3-1}\right)=0$.

In the earlier sections we also proved the following limit:

(5.19) $$\lim_{n\to\infty}\left(x^n\right)=0 \quad \text{provided } |x|<1$$

What is $\lim_{n\to\infty}\left(\left(\frac{1}{5}\right)^n\right)$ *equal to?*

Since $\frac{1}{5}<1$ therefore by the above result (5.19) with $x=\frac{1}{5}$ we have $\lim_{n\to\infty}\left(\left(\frac{1}{5}\right)^n\right)=0$.

Example 18

Evaluate

$$\lim_{n\to\infty}\left(\frac{1}{5n^n}\right)$$

Solution. Since $\left(\frac{1}{5n^n}\right)=\frac{1}{5}\left(\frac{1}{n^n}\right)=\left(\frac{1}{5}\right)\left(\frac{1}{n}\right)^n$ and for $n>1$ we have $\frac{1}{n}<1$ which means that $\lim_{n\to\infty}\left(\left(\frac{1}{n}\right)^n\right)=0$ by (5.19) with $x=\frac{1}{n}$. Therefore

$$\lim_{n\to\infty}\left(\frac{1}{5n^n}\right)=\lim_{n\to\infty}\left(\frac{1}{5}\left(\frac{1}{n}\right)^n\right)=\frac{1}{5}\lim_{n\to\infty}\left(\left(\frac{1}{n}\right)^n\right) \qquad \left[\text{Using } \lim_{n\to\infty}\left(Kx_n\right)=K\lim_{n\to\infty}\left(x_n\right)\right]$$

$$=\frac{1}{5}\times 0=0$$

Hence $\lim_{n\to\infty}\left(\frac{1}{5n^n}\right)=0$.

E2 **Null Sequences**

For the last two examples the limit of the sequence in each case has been **zero**. These are examples of null sequences. A **null sequence** is a real sequence $\left(x_n\right)$ which converges to **zero** as $n\to\infty$, that is $\lim_{n\to\infty}\left(x_n\right)=0$. *Can you think of any other examples of null sequences?*

$$\lim_{n\to\infty}\left(\frac{1}{n}\right)=0,\quad \lim_{n\to\infty}\left(\frac{1}{n^2}\right)=0,\quad \lim_{n\to\infty}\left(\left(x\right)^n\right)=0 \ \text{ provided } \ |x|<1,\ldots$$

Using the propositions of the last section it can be shown that:

If $\lim_{n\to\infty}\left(x_n\right)$ and $\lim_{n\to\infty}\left(y_n\right)$ are null sequences then so are

$$\lim_{n\to\infty}\left(x_n+y_n\right)$$

$$\lim_{n\to\infty}\left(x_n y_n\right)$$

$$\lim_{n\to\infty}\left(Kx_n\right) \text{ where } K \text{ is a constant}$$

You are asked to prove these results in Exercise 5(e).

Example 19

Show that

(5.20) $\lim_{n\to\infty}\left(n^r x^n\right)=0$ where r is a real number and $|x|<1$

Proof. See Exercise 5(e)

We use this general result of Example 19 to evaluate the limit in the next example.

Example 20

Find

$$\lim_{n\to\infty}\left(\frac{2^n-n^6}{n^4-2n+5^n}\right)$$

Solution

What is the dominant term in $\frac{2^n-n^6}{n^4-2n+5^n}$ *?*

5^n is the dominant term. Divide numerator and denominator of $\frac{2^n-n^6}{n^4-2n+5^n}$ by 5^n:

$$\frac{2^n-n^6}{n^4-2n+5^n}=\frac{\frac{2^n}{5^n}-\frac{n^6}{5^n}}{\frac{n^4}{5^n}-\frac{2n}{5^n}+\frac{5^n}{5^n}}$$

$$=\frac{\left(\frac{2}{5}\right)^n-\frac{n^6}{5^n}}{\frac{n^4}{5^n}-\frac{2n}{5^n}+1} \qquad (*)$$

Evaluating each limit:

$$\lim_{n\to\infty}\left(\frac{n^4}{5^n}\right)=\lim_{n\to\infty}\left(n^4\frac{1}{5^n}\right)$$

$$=\lim_{n\to\infty}\left(n^4\left(\frac{1}{5}\right)^n\right)=0 \qquad \left[\text{By (5.20) with } r=4 \text{ and } x=\frac{1}{5}<1\right]$$

Similarly we have $\lim_{n\to\infty}\left(\frac{n^6}{5^n}\right)=0$ $\left[\text{By (5.20) with } r=6 \text{ and } x=\frac{1}{5}<1\right]$.

What is $\lim_{n\to\infty}\left(\frac{2n}{5^n}\right)$ *equal to?*

Consider this term without the constant 2, that is consider $\left(\frac{n}{5^n}\right)$:

$$\lim_{n\to\infty}\left(\frac{n}{5^n}\right)=\lim_{n\to\infty}\left(n\frac{1}{5^n}\right) \qquad \left[\text{Because } \frac{n}{5^n}=n\times\frac{1}{5^n}\right]$$

$$=\lim_{n\to\infty}\left(n^1\left(\frac{1}{5}\right)^n\right)=0 \qquad \left[\text{By (5.20) with } r=1 \text{ and } x=\frac{1}{5}\right]$$

Therefore we have $\lim_{n\to\infty}\left(\frac{2n}{5^n}\right)=2\lim_{n\to\infty}\left(\frac{n}{5^n}\right)=2\times 0=0$.

(5.20) $\lim_{n\to\infty}\left(n^r x^n\right)=0$ where r is a real number and $|x|<1$

What is $\lim_{n\to\infty}\left(\left(\frac{2}{5}\right)^n\right)$ *equal to?*

By (5.19) with $x=\frac{2}{5}<1$ we have $\lim_{n\to\infty}\left(\left(\frac{2}{5}\right)^n\right)=0$.

Substituting all these, $\lim_{n\to\infty}\left(\left(\frac{2}{5}\right)^n\right)=0,\ \lim_{n\to\infty}\left(\frac{n^6}{5^n}\right)=0,\ \lim_{n\to\infty}\left(\frac{n^4}{5^n}\right)=0$ and $\lim_{n\to\infty}\left(\frac{2n}{5^n}\right)=0$,

gives

$$\lim_{n\to\infty}\left(\frac{2^n-n^6}{n^4-2n+5^n}\right)=\lim_{n\to\infty}\left(\frac{\left(\frac{2}{5}\right)^n-\frac{n^6}{5^n}}{\frac{n^4}{5^n}-\frac{2n}{5^n}+1}\right)\qquad[\text{By } (*)]$$

$$=\frac{\lim_{n\to\infty}\left(\left(\frac{2}{5}\right)^n\right)-\lim_{n\to\infty}\left(\frac{n^6}{5^n}\right)}{\lim_{n\to\infty}\left(\frac{n^4}{5^n}\right)-2\lim_{n\to\infty}\left(\frac{n}{5^n}\right)+1}$$

$$=\frac{0-0}{0-0+1}=0\qquad[\text{By the above evaluations}]$$

Hence $\lim_{n\to\infty}\left(\frac{2^n-n^6}{n^4-2n+5^n}\right)=0$ which means it is a null sequence.

Example 21

Evaluate $\lim_{n\to\infty}\left(\sqrt{\frac{n^5-2+5\left(10^n\right)}{n^6+3\left(10^n\right)}}\right)$

Solution

In the last exercise 5(d) we proved that if $\lim_{n\to\infty}(x_n)=L$ then $\lim_{n\to\infty}\left(\sqrt{x_n}\right)=\sqrt{L}$.

What does this result mean?

Means that if the limiting value of the sequence (x_n) is L then the limiting value of $\left(\sqrt{x_n}\right)$ is $\sqrt{L}$. Therefore we first determine $\lim_{n\to\infty}(x_n)$ with $x_n=\frac{n^5-2+5\left(10^n\right)}{n^6+3\left(10^n\right)}$ and then find $\lim_{n\to\infty}\left(\sqrt{x_n}\right)$. *What is* $\lim_{n\to\infty}(x_n)$ *equal to?*

We need to find the dominant term first and then divide numerator and denominator by this dominant term. *What is the dominant term in* $x_n=\frac{n^5-2+5\left(10^n\right)}{n^6+3\left(10^n\right)}$*?*

10^n. Dividing numerator and denominator by 10^n we have

(5.19) $\lim_{n\to\infty}\left(x^n\right)=0$ provided $|x|<1$

$$x_n = \frac{n^5 - 2 + 5\left(10^n\right)}{n^6 + 3\left(10^n\right)} = \frac{\frac{n^5}{10^n} - \frac{2}{10^n} + 5\left(\frac{10^n}{10^n}\right)}{\frac{n^6}{10^n} + 3\left(\frac{10^n}{10^n}\right)}$$

$$= \frac{\frac{n^5}{10^n} - \frac{2}{10^n} + 5}{\frac{n^6}{10^n} + 3} \qquad \left[\text{Because } \frac{10^n}{10^n} = 1\right]$$

To evaluate the limit we consider each component of x_n. *What is* $\lim_{n\to\infty}\left(\frac{n^5}{10^n}\right)$ *equal to?*

Using (5.20) with $r = 5$ and $x = \frac{1}{10}$ we have $\lim_{n\to\infty}\left(\frac{n^5}{10^n}\right) = \lim_{n\to\infty}\left(n^5\left(\frac{1}{10}\right)^n\right) = 0$.

Similarly $\lim_{n\to\infty}\left(\frac{n^6}{10^n}\right) = 0$. *What is* $\lim_{n\to\infty}\left(\frac{2}{10^n}\right)$ *equal to?*

Since by (5.19) with $x = \frac{1}{10}$ we have $\lim_{n\to\infty}\left(\frac{1}{10^n}\right) = \lim_{n\to\infty}\left(\left(\frac{1}{10}\right)^n\right) = 0$ therefore

$$\lim_{n\to\infty}\left(\frac{2}{10^n}\right) = \lim_{n\to\infty}\left(2\frac{1}{10^n}\right) = 2\underbrace{\lim_{n\to\infty}\left(\left(\frac{1}{10}\right)^n\right)}_{=0} = 2\times 0 = 0$$

Substituting these, $\lim_{n\to\infty}\left(\frac{n^5}{10^n}\right) = 0$, $\lim_{n\to\infty}\left(\frac{n^6}{10^n}\right) = 0$ and $\lim_{n\to\infty}\left(\frac{2}{10^n}\right) = 0$, to the limiting value of $x_n = \dfrac{\frac{n^5}{10^n} - \frac{2}{10^n} + 5}{\frac{n^6}{10^n} + 3}$ gives

$$\lim_{n\to\infty}\left(x_n\right) = \frac{\lim_{n\to\infty}\left(\frac{n^5}{10^n}\right) - \lim_{n\to\infty}\left(\frac{2}{10^n}\right) + 5}{\lim_{n\to\infty}\left(\frac{n^6}{10^n}\right) + 3} \qquad \left[\begin{array}{l}\text{The limits of the constants are} \\ \lim_{n\to\infty}(5) = 5 \text{ and } \lim_{n\to\infty}(3) = 3\end{array}\right]$$

$$= \frac{0 - 0 + 5}{0 + 3} = \frac{5}{3}$$

However we need to find $\lim_{n\to\infty}\left(\sqrt{\frac{n^5 - 2 + 5\left(10^n\right)}{n^6 + 3\left(10^n\right)}}\right)$. *How?*

Applying

$$\lim_{n\to\infty}\left(x_n\right) = L \;\Rightarrow\; \lim_{n\to\infty}\left(\sqrt{x_n}\right) = \sqrt{L} \quad \text{with } \lim_{n\to\infty}\left(x_n\right) = \frac{5}{3} \quad \text{we have}$$

(5.19) $\lim_{n\to\infty}\left(x^n\right) = 0$ provided $|x| < 1$

(5.20) $\lim_{n\to\infty}\left(n^r x^n\right) = 0$ where r is a real number and $|x| < 1$

$$\lim_{n\to\infty}\left(\sqrt{x_n}\right)=\sqrt{\frac{5}{3}}$$

Hence $\lim_{n\to\infty}\left(\sqrt{\frac{n^5-2+5\left(10^n\right)}{n^6+3\left(10^n\right)}}\right)=\sqrt{\frac{5}{3}}$.

Example 22

Determine $\lim_{n\to\infty}\left(\left(\frac{4n-1}{n+1}\right)^3\right)$

Solution

How do we find this limit?

We use the proposition proved in question 15 of Exercise 5(d) which says:

If $\lim_{n\to\infty}(x_n)=L$ then $\lim_{n\to\infty}\left((x_n)^m\right)=L^m$ where $m\in\mathbb{N}$. *What does this result mean?*

Means that if the limiting value of the sequence (x_n) is L then the limiting value of $(x_n)^m$ is L^m. We first evaluate $\lim_{n\to\infty}(x_n)$ and then we find $\lim_{n\to\infty}\left((x_n)^m\right)$.

Let $x_n=\frac{4n-1}{n+1}$. *How do we evaluate* $\lim_{n\to\infty}\left(\frac{4n-1}{n+1}\right)$ *?*

Determine the dominant term and then divide the numerator and denominator by this dominant term. *What is the dominant term in* $\frac{4n-1}{n+1}$ *?*

Dominant term is *n*. Dividing numerator and denominator by *n* gives

$$\frac{4n-1}{n+1}=\frac{\frac{4n}{n}-\frac{1}{n}}{\frac{n}{n}+\frac{1}{n}}$$

$$=\frac{4-\frac{1}{n}}{1+\frac{1}{n}}\qquad\text{[Cancelling the } n\text{'s]}$$

Since $\lim_{n\to\infty}\left(\frac{1}{n}\right)=0$ therefore we have

$$\lim_{n\to\infty}\left(\frac{4n-1}{n+1}\right)=\lim_{n\to\infty}\left(\frac{4-\frac{1}{n}}{1+\frac{1}{n}}\right)\qquad\text{[By Above]}$$

$$=\frac{4-\lim_{n\to\infty}\left(\frac{1}{n}\right)}{1+\lim_{n\to\infty}\left(\frac{1}{n}\right)}=\frac{4-0}{1-0}=4$$

Hence applying $\lim_{n\to\infty}(x_n)=L \;\Rightarrow\; \lim_{n\to\infty}\left((x_n)^m\right)=L^m$ with $\lim_{n\to\infty}(x_n)=4$ we have

$$\lim_{n\to\infty}\left(\left(\frac{4n-1}{n+1}\right)^3\right)=4^3=64$$

We can also use some of the inequality propositions of the last section to evaluate limits.

Example 23

Determine $\lim_{n\to\infty}\left(\frac{\cos(n)}{n}\right)$.

Solution

How can we find the given limit, $\lim_{n\to\infty}\left(\frac{\cos(n)}{n}\right)$ *?*

Since the cos function lies between -1 to +1 that is $-1\le\cos(n)\le 1$ we can apply the **sandwich rule** which is stated in Question 12 of Exercise 5(d) as:

If $x_n \le y_n \le z_n$ and $\lim_{n\to\infty}(x_n)=\lim_{n\to\infty}(z_n)=L$ then $\lim_{n\to\infty}(y_n)=L$.

*What does the **sandwich rule** mean?*

This means that if the end limits are equal to L then the limit in the middle is also equal to L.

Because we have $\frac{-1}{n}\le\frac{\cos(n)}{n}\le\frac{1}{n}$ and the end limits are equal to zero,

$\lim_{n\to\infty}\left(\frac{-1}{n}\right)=\lim_{n\to\infty}\left(\frac{1}{n}\right)=0$, therefore by the sandwich rule the middle limit is also zero, which means

$$\lim_{n\to\infty}\left(\frac{\cos(n)}{n}\right)=0$$

SUMMARY

We use propositions of the last section to evaluate limits of sequences.

We have also established general limits which can be used to evaluate other limits:

(5.18) $\lim_{n\to\infty}\left(\frac{K}{n^p}\right)=0$ where p is a positive real number

(5.19) $\lim_{n\to\infty}\left(x^n\right)=0$ provided $|x|<1$

(5.20) $\lim_{n\to\infty}\left(n^r x^n\right)=0$ where r is a real number and $|x|<1$

Generally the first step is to identify the dominant term and then divide the numerator and denominator by this dominant term.