Solution to Diagonalising Matrices

Brief Solution

$\mathbf{P}=\begin{pmatrix}\lambda_1 & \lambda_2\\ 1 & 1\end{pmatrix}$, $\mathbf{P}^{-1}=\dfrac{1}{\lambda_1-\lambda_2}\begin{pmatrix}1 & -\lambda_2\\ -1 & \lambda_1\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$ where $\lambda_1=\dfrac{1+\sqrt{5}}{2}$ and $\lambda_2=\dfrac{1-\sqrt{5}}{2}$.

Complete Solution

We are given that $\mathbf{F}=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$. The eigenvalues are given by

$$\begin{aligned}\det\left(\mathbf{F}-\lambda\mathbf{I}\right)&=\det\begin{pmatrix}1-\lambda & 1\\ 1 & 0-\lambda\end{pmatrix}\\ &=(1-\lambda)(0-\lambda)-1\\ &=\lambda^2-\lambda-1=0\end{aligned}$$

Solving the quadratic by using the formula $\lambda=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ with $a=1,\ b=-1$ and $c=-1$ gives

$$\lambda=\frac{-(-1)\pm\sqrt{(-1)^2-\left[4\times1\times(-1)\right]}}{2}=\frac{1\pm\sqrt{5}}{2}$$

The eigevalues are $\lambda_1=\dfrac{1+\sqrt{5}}{2}$ and $\lambda_2=\dfrac{1-\sqrt{5}}{2}$. *What are the eigenvectors?*

Let **u** be the eigenvector belonging to $\lambda_1=\dfrac{1+\sqrt{5}}{2}$.

$$\begin{aligned}\left[\mathbf{F}-\left(\frac{1+\sqrt{5}}{2}\right)\mathbf{I}\right]\mathbf{u}&=\begin{pmatrix}1-\dfrac{1+\sqrt{5}}{2} & 1\\ 1 & 0-\dfrac{1+\sqrt{5}}{2}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}\\ &=\begin{pmatrix}\dfrac{1-\sqrt{5}}{2} & 1\\ 1 & -\dfrac{1+\sqrt{5}}{2}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}\\ &=\begin{pmatrix}\lambda_2 & 1\\ 1 & -\lambda_1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\qquad\left[\text{Because }\lambda_2=\frac{1-\sqrt{5}}{2}\text{ and }\lambda_1=\frac{1+\sqrt{5}}{2}\right]\end{aligned}$$

Expanding matrices we have

$$\left.\begin{aligned}\lambda_2 x+y&=0\\ x-\lambda_1 y&=0\end{aligned}\right\}\quad\text{gives }x=\lambda_1\text{ and }y=1$$

Note that this is the solution because $\lambda_1\lambda_2 = \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right) = -1$. Thus $\mathbf{u} = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$ is the eigenvector belonging to $\lambda_1 = \frac{1+\sqrt{5}}{2}$. Let $\mathbf{v}$ be the eigenvector belonging to $\lambda_2 = \frac{1-\sqrt{5}}{2}$

$$\left[\mathbf{F} - \left(\frac{1-\sqrt{5}}{2}\right)\mathbf{I}\right]\mathbf{v} = \begin{pmatrix} 1-\frac{1-\sqrt{5}}{2} & 1 \\ 1 & 0-\frac{1-\sqrt{5}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 1 \\ 1 & -\frac{1-\sqrt{5}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & 1 \\ 1 & -\lambda_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Expanding the matrices we have

$$\left.\begin{aligned} \lambda_1 x + y &= 0 \\ x - \lambda_2 y &= 0 \end{aligned}\right\} \quad \text{gives } x = \lambda_2 \text{ and } y = 1$$

Thus $\mathbf{v} = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$ is the eigenvector belonging to $\lambda_2 = \frac{1-\sqrt{5}}{2}$.

Since the eigenvectors **u** and **v** are linearly independent therefore the invertible matrix **P** is given by $\mathbf{P} = (\mathbf{u} \vdots \mathbf{v}) = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}$. *What is the inverse matrix* $\mathbf{P}^{-1}$ *equal to?*

$$\mathbf{P}^{-1} = \frac{1}{\lambda_1 - \lambda_2}\begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix}$$

Substituting $\mathbf{P}^{-1} = \frac{1}{\lambda_1 - \lambda_2}\begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix}$, $\mathbf{F} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{P} = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}$ into $\mathbf{P}^{-1}\mathbf{F}\mathbf{P}$ will give a diagonal matrix with the eigenvalues on the leading diagonal. Let us check this:

$$\mathbf{P}^{-1}\mathbf{F}\mathbf{P} = \frac{1}{\lambda_1 - \lambda_2}\begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{\lambda_1 - \lambda_2}\begin{pmatrix} 1-\lambda_2 & 1 \\ -1+\lambda_1 & -1 \end{pmatrix}\begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{\lambda_1 - \lambda_2}\begin{pmatrix} \lambda_1 - \lambda_1\lambda_2 + 1 & \lambda_2 - (\lambda_2)^2 + 1 \\ -\lambda_1 + (\lambda_1)^2 - 1 & -\lambda_2 + \lambda_1\lambda_2 - 1 \end{pmatrix} \qquad (\dagger)$$

Remember each of the eigenvalues λ_1 and λ_2 satisfy $\lambda^2 - \lambda - 1 = 0$ or $\lambda^2 = \lambda + 1$.
Next we examine each of the entries in the last matrix. Starting with the first entry we have

$$\lambda_1 - \lambda_1\lambda_2 + 1 = (\lambda_1 + 1) - \lambda_1\lambda_2 \quad \text{[Rearranging]}$$
$$= \lambda_1^{\,2} - \lambda_1\lambda_2 \quad \text{[Because } \lambda_1 \text{ satisfies the given eqn } \lambda_1^{\,2} = \lambda_1 + 1\text{]}$$
$$= \lambda_1(\lambda_1 - \lambda_2) \quad \text{[Factorising]}$$

Similarly for the last entry we have

$$-\lambda_2 + \lambda_1\lambda_2 - 1 = -(\lambda_2 + 1 - \lambda_1\lambda_2) \quad \text{[Rearranging]}$$
$$= -(\lambda_2^{\,2} - \lambda_1\lambda_2) \quad \text{[Because } \lambda_2 \text{ satisfies the given eqn } \lambda_2^{\,2} = \lambda_2 + 1\text{]}$$
$$= \lambda_2(\lambda_1 - \lambda_2) \quad \text{[Factorising and taking in the negative sign]}$$

Both the other two entries satisfy the quadratic equation $\lambda^2 - \lambda - 1 = 0$.

$$-\lambda_1 + (\lambda_1)^2 - 1 = (\lambda_1)^2 - \lambda_1 - 1 = 0$$
$$\lambda_2 - (\lambda_2)^2 + 1 = -\left[(\lambda_2)^2 - \lambda_2 - 1\right] = 0$$

Substituting these $\lambda_1 - \lambda_1\lambda_2 + 1 = \lambda_1(\lambda_1 - \lambda_2)$, $-\lambda_1 + (\lambda_1)^2 - 1 = 0$, $\lambda_2 - (\lambda_2)^2 + 1 = 0$ and $-\lambda_2 + \lambda_1\lambda_2 - 1 = \lambda_2(\lambda_1 - \lambda_2)$ into (†) gives

$$\mathbf{P}^{-1}\mathbf{FP} = \frac{1}{\lambda_1 - \lambda_2}\begin{pmatrix} \lambda_1(\lambda_1 - \lambda_2) & 0 \\ 0 & \lambda_2(\lambda_1 - \lambda_2) \end{pmatrix}$$
$$= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \left[\text{Scalar multiplying } \frac{1}{\lambda_1 - \lambda_2}\right]$$