Chapter 2 : Infinite Series

Section **B Telescoping and Harmonic Series**

By the end of this section you will be able to

- use partial fractions to test for convergence and determine the sum of a series
- prove the divergence of the harmonic series

B1 **Telescoping Series**

In this subsection we use partial fractions to test a series for convergence and find its sum. You will need to revise how to write a given fraction into its partial fraction decomposition and the procedure involved in finding the unknowns. Remember from your previous studies in this topic you always have unknowns in the initial partial fraction decomposition and normally these are denoted by (capital) letters. Before going through the rest of this section make sure you can convert a given fraction into its partial fractions.

Another technique that we use throughout this section is grouping and rearranging the partial sum, S_n. For example we can remove brackets:

$$\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots+\frac{1}{n}-\frac{1}{n+1}$$

or we can add brackets:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots+\frac{1}{n}-\frac{1}{n+1}=\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

We can also rearrange the order of the sum as the following example shows:

$$\frac{1}{2}+1+\frac{1}{4}+\frac{1}{3}+\frac{1}{6}+\frac{1}{5}+\ldots+\frac{1}{n}+\frac{1}{n-1}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots+\frac{1}{n-1}+\frac{1}{n}$$

Also we will need to use our results on sequences from the last chapter such as

(2.4) $$\lim_{n\to\infty}\left(\frac{1}{n}\right)=0$$

Example 7

Test the following series

$$\sum_{k=1}^{\infty}\left(\frac{1}{k(k+1)}\right)$$

for convergence. If it converges determine its sum.

Solution

Let S_n be the nth partial sum of the series. *How do we write* S_n *?*

S_n is the sum of the first n terms so we have

(*) $$S_n=\sum_{k=1}^{n}\left(\frac{1}{k(k+1)}\right)=\frac{1}{1(2)}+\frac{1}{2(3)}+\frac{1}{3(4)}+\ldots+\frac{1}{n(n+1)}$$

Clearly it is not a geometric series or any other series that we have covered in Section A. *So how do we test this for convergence?*

We rewrite the sum in a different format by using partial fractions. We convert the general term, $\frac{1}{k(k+1)}$, into partial fractions. *How?*

Using the standard form for partial fractions we have the identity:

$$\frac{1}{k(k+1)} = \frac{A}{k} + \frac{B}{k+1} \qquad (\dagger)$$

What do we need to find?

Values of the unknowns A and B. We multiply both sides of (†) by $k(k+1)$ to determine A and B.

$$1 = A(k+1) + B(k) \qquad (\dagger\dagger)$$

Since this (††) is an identity we can substitute any values of k. We substitute $k=-1$ into (††). *Why?*

Because the A term becomes zero and so we can find B:

$$1 = A\underbrace{(-1+1)}_{=0} + B(-1) \quad \text{which gives} \quad 1 = -B \quad \Rightarrow \quad B = -1$$

Similarly substituting $k=0$ into (††):

$$1 = A(0+1) + B(0) \quad \text{which gives} \quad A = 1$$

Putting $A=1$ and $B=-1$ into (†) yields the partial fractions:

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

Substituting this into (*) gives

$$S_n = \sum_{k=1}^{n}\left(\frac{1}{k(k+1)}\right) = \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$= \underbrace{\left(1-\frac{1}{2}\right)}_{\text{Putting } k=1} + \underbrace{\left(\frac{1}{2}-\frac{1}{3}\right)}_{\text{Putting } k=2} + \underbrace{\left(\frac{1}{3}-\frac{1}{4}\right)}_{\text{Putting } k=3} + \ldots \underbrace{-\frac{1}{n}\Big)}_{\text{Putting } k=n-1} + \underbrace{\left(\frac{1}{n}-\frac{1}{n+1}\right)}_{\text{Putting } k=n}$$

$$= 1 \underbrace{-\frac{1}{2}+\frac{1}{2}}_{=0} \underbrace{-\frac{1}{3}+\frac{1}{3}}_{=0} \underbrace{-\frac{1}{4}+\frac{1}{4}}_{=0} + \ldots \underbrace{-\frac{1}{n}+\frac{1}{n}}_{=0} - \frac{1}{n+1} \qquad \left[\begin{array}{l}\text{Removing}\\ \text{Brackets}\end{array}\right]$$

$$S_n = 1 - \frac{1}{n+1} \qquad \left[\begin{array}{l}\text{Because ALL the other}\\ \text{terms add upto 0}\end{array}\right]$$

Hence for the given infinite series we have

$$\sum_{k=1}^{\infty}\left(\frac{1}{k(k+1)}\right) = \lim_{n\to\infty}(S_n)$$

$$= \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right) \qquad \left[\text{Substituting } S_n = 1-\frac{1}{n+1}\right]$$

$$= 1 - \lim_{n\to\infty}\left(\frac{1}{n+1}\right) = 1 - 0 = 1 \qquad [\text{By } (2.1)]$$

The series $\sum_{k=1}^{\infty}\left(\frac{1}{k(k+1)}\right) = 1$. Hence the given series converges with sum 1.

(2.4) $$\lim_{n\to\infty}(1/n) = 0$$

The above is an example of a **telescoping** series. A series whose terms cancel leaving just a few terms is called a telescoping series.
The next example is another telescoping series and the procedure for determining the convergence and sum is similar to Example 7. The only difficulty in Example 8 is realizing nearly all the terms cancel.

Example 8
Test
$$\sum_{k=2}^{\infty}\left(\frac{1}{k^2-1}\right)$$
for convergence. If the series does converge then determine its sum.
Solution
Writing out the series informally we have
$$\sum_{k=2}^{\infty}\left(\frac{1}{k^2-1}\right)=\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\ldots$$
How can we test this series for convergence?
Again we can use partial fractions to rewrite the general term, $\frac{1}{k^2-1}$. *How?*
We first factorize the denominator, k^2-1. *What does this factorize into?*
$$k^2-1=(k-1)(k+1) \qquad \left[\text{Difference of Two Squares}\right]$$
We have
$$\frac{1}{k^2-1}=\frac{1}{(k-1)(k+1)}$$
From our previous knowledge of partial fractions we can convert the Right Hand Side into the following identity:
$$\frac{1}{(k-1)(k+1)}=\frac{A}{k-1}+\frac{B}{k+1} \qquad (\dagger)$$
How do we find the values of the unknowns A *and* B*?*
Multiplying both sides of (†) by $(k-1)(k+1)$ gives the identity
$$1=A(k+1)+B(k-1) \qquad (\dagger\dagger)$$
What values of k *do we substitute into (††)?*
Putting $k=1$ into (††):
$$1=A(1+1)+B\underbrace{(1-1)}_{=0}$$
$$1=2A \text{ which gives } A=\frac{1}{2}$$
Substituting $k=-1$ into (††):
$$1=A\underbrace{(-1+1)}_{=0}+B(-1-1)$$
$$1=-2B \text{ which gives } B=-\frac{1}{2}$$
Putting $A=\frac{1}{2}$ and $B=-\frac{1}{2}$ into (†) yields

$$\frac{1}{(k-1)(k+1)}=\frac{1}{2(k-1)}-\frac{1}{2(k+1)}$$
$$=\frac{1}{2}\left[\frac{1}{k-1}-\frac{1}{k+1}\right] \qquad \begin{pmatrix}\text{Taking out a Common}\\ \text{Factor of } 1/2\end{pmatrix}$$

We have written the partial fractions decomposition of the general term in the series, that is

$$\frac{1}{k^2-1}=\frac{1}{(k-1)(k+1)}=\frac{1}{2}\left[\frac{1}{k-1}-\frac{1}{k+1}\right]$$

But how can we test the given series, $\sum_{k=2}^{\infty}\left(\frac{1}{k^2-1}\right)$*, for convergence?*

We first examine the nth partial sum, S_n, which is the sum of the first n terms.

$$S_n=\sum_{k=2}^{n}\left(\frac{1}{k^2-1}\right)=\sum_{k=2}^{n}\frac{1}{2}\left(\frac{1}{k-1}-\frac{1}{k+1}\right) \qquad \left[\text{Because } \frac{1}{k^2-1}=\frac{1}{2}\left(\frac{1}{k-1}-\frac{1}{k+1}\right)\right]$$
$$=\frac{1}{2}\sum_{k=2}^{n}\left(\frac{1}{k-1}-\frac{1}{k+1}\right) \qquad \left[\text{Taking out } \frac{1}{2}\right]$$
$$=\frac{1}{2}\left[\underbrace{\left(\frac{1}{1}-\frac{1}{3}\right)}_{\text{Putting } k=2}+\underbrace{\left(\frac{1}{2}-\frac{1}{4}\right)}_{\text{Putting } k=3}+\underbrace{\left(\frac{1}{3}-\frac{1}{5}\right)}_{\text{Putting } k=4}+\ldots+\underbrace{\left(\frac{1}{n-2}-\frac{1}{n}\right)}_{\text{Putting } k=n-1}+\underbrace{\left(\frac{1}{n-1}-\frac{1}{n+1}\right)}_{\text{Putting } k=n}\right]$$
$$=\frac{1}{2}\left(\underbrace{1+\frac{1}{2}}_{=3/2}\underbrace{-\frac{1}{3}+\frac{1}{3}}_{=0}\underbrace{-\frac{1}{4}+\frac{1}{4}}_{=0}\underbrace{+\ldots-}_{=0}\ \frac{1}{n}-\frac{1}{n+1}\right) \qquad \begin{bmatrix}\text{Rearranging}\\ \text{Sum}\end{bmatrix}$$
$$S_n=\frac{1}{2}\left(\frac{3}{2}-\frac{1}{n}-\frac{1}{n+1}\right) \qquad [\text{Because All the other Terms give } 0]$$

It is difficult to see why nearly all the terms vanish but try a few values of n for yourself in the sum. Hence for the given infinite series we have

$$\sum_{k=2}^{\infty}\left(\frac{1}{k^2-1}\right)=\lim_{n\to\infty}(S_n)$$
$$=\lim_{n\to\infty}\left[\frac{1}{2}\left(\frac{3}{2}-\frac{1}{n}-\frac{1}{n+1}\right)\right] \qquad [\text{Substituting for } S_n]$$
$$=\frac{1}{2}\left[\frac{3}{2}-\lim_{n\to\infty}\left(\frac{1}{n}\right)-\lim_{n\to\infty}\left(\frac{1}{n+1}\right)\right]=\frac{1}{2}\left[\frac{3}{2}-0-0\right]=\frac{3}{4} \qquad [\text{By } (2.1)]$$

Therefore the infinite series $\sum_{k=2}^{\infty}\left(\frac{1}{k^2-1}\right)$ converges with sum $\frac{3}{4}$.

As stated earlier, $\sum_{k=2}^{\infty}\left(\frac{1}{k^2-1}\right)$ is another example of a telescoping series because nearly all the terms cancel leaving just a few end terms.

(2.4) $$\lim_{n\to\infty}(1/n)=0$$

B2 **Harmonic Series**

In this subsection we investigate an important series called the **harmonic** series which is defined by

$$\sum_{k=1}^{\infty}\left(\frac{1}{k}\right)$$

Writing out the infinite series we have

$$\sum_{k=1}^{\infty}\left(\frac{1}{k}\right)=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots$$

We show that this series diverges in the next example. This is not an easy task because we examine the 2^n th partial sums rather than the nth partial sum. The difficulty is how would we know to look at the 2^n th partial sums if we were not told. The other difficulty is we need to apply the following proposition from the last chapter:

Proposition (1.??). A convergent sequence of real numbers is bounded.

Remember this proposition says that a convergent sequence must be bounded. So if a sequence is not bounded then it is a **divergent** sequence. In Example 9 we show that the sequence of partial sums, S_n, is unbounded therefore $\lim_{n\to\infty}(S_n)$ is divergent.

We also need to use inequality results to show divergence of the harmonic series. Try proving the divergence of the harmonic series for yourself before going through the proof below. This is always a good way of understanding a proof. Of course you might come up with a much more elegant proof than the one in Example 9.

Example 9

Prove that the harmonic series

$$\sum_{k=1}^{\infty}\left(\frac{1}{k}\right)$$

is divergent.

Proof. Let S_n be the nth partial sum. *How do we write S_n?*

It is the sum of the first n terms of the given series:

$$S_n=\sum_{k=1}^{n}\left(\frac{1}{k}\right)=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}$$

What do we need to prove?

We are required to prove that $\lim_{n\to\infty}(S_n)$ diverges.

We first investigate the partial sums S_{2^n}. Writing down the partial sum for each power of 2 that is 2^n and using inequalities we have

For $2^0=1$: $S_1=1$ [Sum of the First Term]

For $2^1=2$: $S_2=1+\frac{1}{2}=\frac{3}{2}$ [Sum of the First 2 Terms]

For $2^2=4$:

$$S_4=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right) \qquad \left[\text{Sum of the First 4 Terms}\right]$$

$$>1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)=2 \qquad \left[\text{Because } \frac{1}{3}>\frac{1}{4}\right]$$

For $2^3=8$:

$$S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \qquad \left[\text{Sum of the First 8 Terms}\right]$$

$$= \underbrace{\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)}_{=S_4 > 2 \text{ by above}} + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \qquad \left[\text{Bracketing Terms}\right]$$

$$> 2 + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$

$$> 2 + \underbrace{\left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)}_{=4/8} \qquad \left[\text{Because } \frac{1}{5} > \frac{1}{6} > \frac{1}{7} > \frac{1}{8}\right]$$

$$= 2 + \frac{1}{2} = \frac{5}{2}$$

For $2^4 = 16$:

$$S_{16} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} \qquad \left[\begin{array}{l}\text{Sum of the} \\ \text{First 16 Terms}\end{array}\right]$$

$$= \underbrace{\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{8}\right)}_{=S_8 > \frac{5}{2} \text{ by above}} + \left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}\right) \qquad \left[\text{Bracketing Terms}\right]$$

$$> \frac{5}{2} + \left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}\right)$$

$$> \frac{5}{2} + \underbrace{\left(\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}\right)}_{8 \text{ Copies}} \qquad \left[\text{Because } \frac{1}{9} > \frac{1}{10} > \frac{1}{11} > \ldots > \frac{1}{15} > \frac{1}{16}\right]$$

$$= \frac{5}{2} + \frac{8}{16} = \frac{5}{2} + \frac{1}{2} = \frac{6}{2}$$

Summarizing the above the sums, S_{2^n}, we have

$$S_1 = S_{2^0} = 1$$

$$S_2 = S_{2^1} = \frac{3}{2}$$

$$S_4 = S_{2^2} > \frac{4}{2}$$

$$S_8 = S_{2^3} > \frac{5}{2}$$

$$S_{16} = S_{2^4} > \frac{6}{2}$$

What is the inequality relationship between S_{2^n} and the answer on the Right Hand Side in these sums?

$$S_{2^n} \geq \frac{n+2}{2}$$

Since $\frac{n+2}{2}$ has no limit as $n \to \infty$ therefore $\lim\limits_{n\to\infty}\left(\frac{n+2}{2}\right)$ is unbounded.

Hence the partial sums $S_1,\ S_2,\ S_4,\ S_8,\ S_{16},\ ...,\ S_{2^n},\ ...$ increase without bound. This means that (S_n) is unbounded which implies that (S_n) is divergent by proposition (1.??). That is $\lim_{n\to\infty}(S_n)$ diverges. We have proven that the harmonic series, $\sum_{k=1}^{\infty}\left(\frac{1}{k}\right)$, diverges.

■

SUMMARY
We can evaluate sums of infinite series by using partial fractions.
Partial fractions can be applied for evaluating the sum of a **telescoping** series which is a series where nearly all the terms cancel.
Harmonic series is defined as

$$\sum_{k=1}^{\infty}\left(\frac{1}{k}\right)=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+...$$

and is a divergent series.