Tough Nut to Crack – Trigonometry Problem I:

Problem

Let x and y be real numbers such that

$$\sin(x)+\sin(y)=a$$

$$\cos(x)+\cos(y)=b$$

Determine $\sin(x+y)$ and $\cos(x+y)$.

Solution

By applying the trigonometric identities we have

$$\sin(x)+\sin(y)=2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos(x)+\cos(y)=2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

Dividing these equations yields

$$\frac{\sin(x)+\sin(y)}{\cos(x)+\cos(y)}=\frac{2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}{2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}$$

$$=\tan\left(\frac{x+y}{2}\right)=\frac{a}{b}$$

Let $\alpha=x+y$ and $t=\tan\left(\frac{\alpha}{2}\right)$ then

$$\sin(\alpha)=\frac{2t}{1+t^2}=\frac{2a/b}{1+\frac{a^2}{b^2}}=\frac{2ab}{b^2+a^2}$$

$$\cos(\alpha)=\frac{1-t^2}{1+t^2}=\frac{1-\frac{a^2}{b^2}}{1+\frac{a^2}{b^2}}=\frac{b^2-a^2}{b^2+a^2}$$

Substituting back $\alpha=x+y$ gives us our result:

$$\sin(x+y)=\frac{2ab}{b^2+a^2}$$

$$\cos(x+y)=\frac{b^2-a^2}{b^2+a^2}$$