Complete Solutions to Exercise 1h

1. We need to prove 9 divides $10^n - 1$ for any natural number *n*.
Proof.
<u>Step 1</u> The result is true for $n = 1$.
<u>Step 2</u> Assume the result holds for $n = k$:
$$10^k - 1 = 9m \qquad (*)$$
where *m* is an integer.
<u>Step 3</u> Required to prove the given proposition holds for $n = k + 1$:
$$10^{k+1} - 1 = 9 \times (\text{integer})$$
Examining the LHS we have
$$\begin{aligned} 10^{k+1} - 1 &= 10\left(10^k\right) - 1 \\ &= 9\left(10^k\right) + \underbrace{10^k - 1}_{=9m \text{ by } (*)} \qquad [\text{Rewriting } 10 = 9 + 1] \\ &= 9\left(10^k\right) + 9m = 9\left(10^k + m\right) \end{aligned}$$
Hence $10^{k+1} - 1$ is $9 \times (\text{integer})$ so 9 is a divisor of $10^{k+1} - 1$. By mathematical induction we have our result, 9 divides $10^n - 1$ for all $n \in$.
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2. *How do we prove* $3 \mid \left(n^3 - n\right)$ *?*
Using mathematical induction.
Proof.
The result holds for $n = 1$ because $n^3 - n = 1^3 - 1 = 0$ and $3 \times 0 = 0$ so 3 divides $n^3 - n$ for $n = 1$.
Assume the result is true for $n = k$. This means there is an integer *m* such that
$$3m = k^3 - k \qquad (\dagger)$$
Need to prove $3 \mid \left((k+1)^3 - (k+1)\right)$. Expanding the RHS of this gives
$$\begin{aligned} (k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= \underbrace{k^3 - k}_{=3m \text{ by } (\dagger)} + 3k^2 + 3k = 3\left(m + k^2 + k\right) \end{aligned}$$
Hence 3 divides $(k+1)^3 - (k+1)$ which means the result holds for $n = k + 1$. By mathematical induction we have our result.
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3. *Proof.*
Let $P(n)$ be the given proposition. Check $P(1)$:
$$3 \mid 1(1+1)(1+2) \quad ✓$$
Assume the result is true for $P(k)$. There exists an integer *m* such that
$$k(k+1)(k+2) = 3m \qquad (*)$$
Using this prove $P(k+1)$, that is required to prove
$$3 \times (\text{integer}) = (k+1)(k+2)(k+3)$$

Expanding the last bracket of the RHS:

$$\underbrace{k(k+1)(k+2)}_{=3m}+3(k+1)(k+2)=3m+3(k+1)(k+2)$$

Hence this is a multiple of 3. Therefore we have proven $P(k+1)$ so by mathematical induction we have our result.

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4. To show n^2-n is an even number we just show by mathematical induction that $2 \mid (n^2-n)$.

5. *Proof*. We first check the proposition for $n=1$:

$$a=\frac{a(1-r)}{1-r}=a \qquad \text{[Cancelling } (1-r)\text{'s]}$$

Hence the proposition is true for $n=1$. *What is our next step?*
Assume the proposition is true for $n=k$, that is

$$a+ar+ar^2+\ldots+ar^{k-1}=\frac{a(1-r^k)}{1-r} \qquad (\$)$$

We need to prove the proposition for $n=k+1$ which is the following;

$$a+ar+ar^2+\ldots+ar^{k-1}+ar^k=\frac{a(1-r^{k+1})}{1-r} \qquad (\#)$$

What do we need to prove?
Left Hand Side is equa1 to the Right Hand side of (#). Examining the Left Hand Side of $(\#)$ and using ($) we have

$$\begin{aligned}
a+ar+ar^2+\ldots+ar^{k-1}+ar^k &= \underbrace{a+ar+ar^2+\ldots+ar^{k-1}}_{=\frac{a(1-r^k)}{1-r}\text{ by (\$)}}+ar^k \\
&= \frac{a(1-r^k)}{1-r}+ar^k \\
&= \frac{a(1-r^k)+ar^k(1-r)}{1-r} && \text{[Common denominator]} \\
&= \frac{a-ar^k+ar^k-ar^kr}{1-r} && \text{[Expanding brackets on numerator]} \\
&= \frac{a-ar^{k+1}}{1-r} && \text{[Because } -ar^k+ar^k=0\text{]} \\
&= \frac{a(1-r^{k+1})}{1-r} && \text{[Factorizing numerator]}
\end{aligned}$$

The last line is the Right Hand Side of (#). Therefore we have shown Left Hand Side is equa1 to the Right Hand side of (#). Hence we have our result.

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6. *Proof.*
Clearly the result is true for $n=1$ because we are given $a \mid b$.
Assume for any natural number k we have

$$a^k \mid b^k \qquad (*)$$

Need to prove that $a^{k+1} \mid b^{k+1}$. Examining this we have

$$a^{k+1} = a^k a$$

By (*) $a^k \mid b^k$ and by the initial step $a \mid b$ therefore

$$a^k a \mid b^k b$$

Hence we have our result $a^{k+1} \mid b^{k+1}$. This completes our proof.

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7. We cannot assume the result for $n = k+1$:

$$k+1 \geq 2^{k+1} - 1 \qquad (**)$$

Remember induction says $P(k) \Rightarrow P(k+1)$.

8. (i) We need to prove $3n^2 + 3n + 1 < 2n^3$ for $n \geq 3$.

Proof.

The result is true for $n = 3$ because

$$3(3)^2 + 3(3) + 1 = 37 < 2(3)^3 = 54 \quad \checkmark$$

Assume the proposition is true for $n = k$:

$$3k^2 + 3k + 1 < 2k^3 \qquad (*)$$

Consider the proposition for the next number $n = k+1$. Required to prove

$$3(k+1)^2 + 3(k+1) + 1 < 2(k+1)^3 \qquad (\dagger)$$

Expanding the LHS of (†) gives

$$\begin{aligned} 3(k+1)^2 + 3(k+1) + 1 &= 3(k^2 + 2k + 1) + 3k + 3 + 1 \\ &= 3k^2 + 6k + 3 + 3k + 3 + 1 \\ &= \underbrace{3k^2 + 3k + 1}_{<2k^3 \text{ by } (*)} + 6k + 6 < 2k^3 + 6k + 6 \end{aligned}$$

Expanding the RHS of (†) gives

$$\begin{aligned} 2(k+1)^3 &= 2(k^3 + 3k^2 + 3k + 1) \\ &= 2k^3 + 6k^2 + 6k + 2 \end{aligned}$$

Now the LHS $\leq$ RHS $\Leftrightarrow$ $2k^3 + 6k + 6 < 2k^3 + 6k^2 + 6k + 2$ $\Leftrightarrow$ $4 < 6k^2$. Since k is a natural number so $4 < 6k^2$. Hence the inequality in (†) has been established so by mathematical induction we have our given result.

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(ii) We need to prove $3^n > n^3$ for $n \geq 4$.

Proof.

Check is true for $n = 4$:

$$3^4 = 81 > 4^3 = 64 \quad \checkmark$$

Assume the proposition is true $n = k$, that is

$$3^k > k^3 \qquad (*)$$

Required to prove the result for $n = k+1$, that is

$$3^{k+1} > (k+1)^3 \qquad (**)$$

Examining the LHS we have

$$3^{k+1} = 3^k 3 > 3k^3 \qquad [\text{By } (*)]$$

Expanding the RHS of (**) gives

$$(k+1)^3 = k^3 + 3k^2 + 3k + 1 \qquad (\dagger)$$

By result of part (i) we have

$$3k^2 + 3k + 1 < 2k^3$$

Substituting this into (†) gives

$$(k+1)^3 = k^3 + \underbrace{3k^2 + 3k + 1}_{<2k^3} < k^3 + 2k^3 = 3k^3$$

Hence $(k+1)^3 < 3k^3 < 3^k 3 = 3^{k+1}$. This is the inequality in (**) so by mathematical induction we have proven that $3^n > n^3$.

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9. (i) You should be able to verify this result.
(ii) Required to prove $2^n \ge n^2$ for $n \ge 4$.
Proof.
Clearly the result is true for $n = 4$ because

$$2^4 = 16 \ge 4^2 = 16$$

Assume the result is true for $n = k$, that is

$$2^k \ge k^2 \qquad (*)$$

Required to prove

$$2^{k+1} \ge (k+1)^2 \qquad (**)$$

Consider the LHS of this inequality:

$$2^{k+1} = 2^k 2 \ge 2k^2 \qquad [\text{By } (*)]$$

We have

$$2k^2 \ge (k+1)^2 \quad \Leftrightarrow \quad 2k^2 \ge k^2 + 2k + 1 \quad \Leftrightarrow \quad k^2 \ge 2k + 1$$

By result of part (i) we have $k^2 \ge 2k + 1$. Hence we have $2^{k+1} \ge 2k^2 \ge (k+1)^2$ which is (**). By mathematical induction we have our required result.

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10. We need to prove Bernoulli's inequality;

$$(1+x)^n \ge 1 + nx \quad \text{where } x > -1$$

Proof.
Check the result for $n = 1$:

$$(1+x)^1 = 1 + x \ge 1 + x$$

The result holds for $n = 1$. Assume the result is true for $n = k$:

$$(1+x)^k \ge 1 + kx \qquad (\dagger)$$

Required to prove

$$(1+x)^{k+1} \ge 1 + (k+1)x$$

Using the rules of indices on the LHS:

$$\begin{aligned}(1+x)^{k+1} &= (1+x)^k(1+x)\\ &\geq \underbrace{(1+kx)}_{\text{by } (\dagger)}(1+x)\\ &= 1+(k+1)x+kx^2 \geq 1+(k+1)x \qquad \left[\text{Because } kx^2 \geq 0\right]\end{aligned}$$

Hence the result holds for $n = k+1$. By mathematical induction we have Bernoulli's inequality.

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11. *Proof*. By applying mathematical induction we have:
Check the result is true for $n = 1$, that is

$$\sin(x) = \frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2(1)+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)}$$

$$= \frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{3}{2}x\right)}{2\sin\left(\frac{x}{2}\right)} \qquad (\dagger)$$

How do we show the Right Hand Side simplifies to $\sin(x)$ *?*

We need to use the trigonometric identity:

$$\cos(A)-\cos(B) = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

on the numerator of (†).

$$\begin{aligned}\cos\left(\frac{x}{2}\right)-\cos\left(\frac{3x}{2}\right) &= -2\sin\left(\frac{x+3x}{4}\right)\sin\left(\frac{x-3x}{4}\right)\\ &= -2\sin(x)\sin\left(-\frac{x}{2}\right) \qquad [\text{Simplifying}]\\ &= -2\sin(x)\left(-\sin\left(\frac{x}{2}\right)\right) \qquad \left[\text{Because } \sin(-\,\bar{}\,) = -\sin(\,\bar{}\,)\right]\\ &= 2\sin(x)\sin\left(\frac{x}{2}\right)\end{aligned}$$

Substituting this into (†) gives

$$\sin(x) = \frac{2\sin(x)\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)} = \sin(x) \qquad \left[\text{Cancelling } 2\sin\left(\frac{x}{2}\right)\right]$$

Hence the proposition is true for $n = 1$. Next we assume the proposition is true for $n = k$:

$$\sin(x)+\sin(2x)+\ldots+\sin(kx) = \frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2k+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)} \qquad (*)$$

We need to prove the proposition for $n = k+1$, that is

$$\sin(x)+\sin(2x)+...+\sin(kx)+\sin\left(\left(k+1\right)x\right)=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2\left(k+1\right)+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)}$$

$$=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\left(\frac{2k+3}{2}\right)x\right)}{2\sin\left(\frac{x}{2}\right)} \qquad (**)$$

What do we need to show?

The Left Hand Side is equal to the Right Hand Side of (**). Let's examine the Left Hand Side first.

$$\underbrace{\sin(x)+\sin(2x)+...+\sin(kx)}_{=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2k+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)} \text{ by (*)}}+\sin\left(\left(k+1\right)x\right)=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2k+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)}+\sin\left(\left(k+1\right)x\right)$$

$$=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2k+1}{2}x\right)+2\sin\left(\frac{x}{2}\right)\sin\left(\left(k+1\right)x\right)}{2\sin\left(\frac{x}{2}\right)}$$

[Common denominator]

What do we do next?

We can use the following trigonometric identity on the last term of the numerator:

$$\sin\left(A\right)\sin\left(B\right)=\frac{1}{2}\left[\cos\left(A-B\right)-\cos\left(A+B\right)\right]$$

We have

$$2\sin\left(\frac{x}{2}\right)\sin\left(\left(k+1\right)x\right)=\left[\cos\left(\frac{x}{2}-\left(k+1\right)x\right)-\cos\left(\frac{x}{2}+\left(k+1\right)x\right)\right]$$

$$=\left[\cos\left(\frac{x}{2}-\frac{\left(2k+2\right)x}{2}\right)-\cos\left(\frac{x}{2}+\frac{\left(2k+2\right)x}{2}\right)\right]$$

$$=\left[\cos\left(\frac{x-2kx-2x}{2}\right)-\cos\left(\frac{x+2kx+2x}{2}\right)\right]$$

$$=\left[\cos\left(\frac{-x-2kx}{2}\right)-\cos\left(\frac{3x+2kx}{2}\right)\right]$$

$$=\left[\cos\left(\frac{x+2kx}{2}\right)-\cos\left(\frac{3x+2kx}{2}\right)\right] \qquad \left[\text{Using }\cos\left(-^{-}\right)=\cos\left(^{-}\right)\right]$$

$$=\left[\cos\left(\frac{\left(2k+1\right)x}{2}\right)-\cos\left(\frac{\left(2k+3\right)x}{2}\right)\right]$$

Substituting this into the above we have

$$\sin(x)+\sin(2x)+...+\sin\left(\left(k+1\right)x\right)=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{2k+1}{2}x\right)+\left[\cos\left(\frac{\left(2k+1\right)x}{2}\right)-\cos\left(\frac{\left(2k+3\right)x}{2}\right)\right]}{2\sin\left(\frac{x}{2}\right)}$$

$$=\frac{\cos\left(\frac{x}{2}\right)-\cos\left(\frac{\left(2k+3\right)x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}$$

$\left[\text{Because } -\cos\left(\frac{2k+1}{2}x\right)+\cos\left(\frac{\left(2k+1\right)x}{2}\right)=0\right]$. Hence we have the Right Hand Side of (**). Therefore we have our required result and the proposition is proved by induction.

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12. *Proof*. We first check the proposition for $n=1$

$$(a+b)^1=a^1+b^1=a+b$$

Hence the proposition is true for $n=1$. *What is our next step?*
Assume the proposition is true for $n=k$, that is

$$\left(a+b\right)^k=a^k+ka^{k-1}b+\frac{k\left(k-1\right)}{2!}a^{k-2}b^2+...+b^k \qquad (*)$$

We need to prove the proposition for $n=k+1$ which is the following;

$$\left(a+b\right)^{k+1}=a^{k+1}+\left(k+1\right)a^{k-1+1}b+\frac{\left(k+1\right)\left(\left(k+1\right)-1\right)}{2!}a^{(k+1)-2}b^2+...+b^{k+1}$$

$$=a^{k+1}+\left(k+1\right)a^kb+\frac{\left(k+1\right)k}{2!}a^{k-1}b^2+...+b^{k+1}$$

What do we need to show to prove this?
Left Hand Side is equa1 to the Right Hand Side. *How?*
Using (*) and algebraic manipulation.

$$(a+b)^{k+1} = (a+b)^k (a+b)^1$$

$$= \left(\underbrace{a^k + ka^{k-1}b + \frac{k(k-1)}{2!}a^{k-2}b^2 + \ldots + b^k}_{\text{by (*)}} \right)(a+b)$$

$$= \underbrace{a^k a + ka^{k-1}ba + \frac{k(k-1)}{2!}a^{k-2}b^2 a + \ldots + b^k a}_{\text{Multiplying the long bracket by } a} +$$

$$\underbrace{a^k b + ka^{k-1}bb + \frac{k(k-1)}{2!}a^{k-2}b^2 b + \ldots + b^k b}_{\text{Multiplying the long bracket by } b}$$

$$= a^{k+1} + ka^k b + \frac{k(k-1)}{2!}a^{k-1}b^2 + \ldots + ab^k +$$

$$a^k b + ka^{k-1}b^2 + \frac{k(k-1)}{2!}a^{k-2}b^3 + \ldots + b^{k+1}$$

[Simplifying by using rules of indices]

$$= a^{k+1} + (k+1)a^k b + \left[\frac{k(k-1)}{2!} + k\right]a^{k-1}b^2 + \ldots + b^{k+1} \qquad \left[\begin{array}{l}\text{Collecting}\\ \text{like terms}\end{array}\right]$$

$$= a^{k+1} + (k+1)a^k b + \underbrace{\left[\frac{k(k+1)}{2!}\right]}_{\text{because } \frac{k(k-1)}{2!}+k=\frac{k(k+1)}{2!}} a^{k-1}b^2 + \ldots + b^{k+1}$$

Hence we have

$$(a+b)^{k+1} = a^{k+1} + (k+1)a^k b + \frac{(k+1)k}{2!}a^{k-1}b^2 + \ldots + b^{k+1}$$

The required result. We have proven the binomial theorem for all natural numbers.

13. *Proof.*

Check for $n=1$; that is $p_1 = 2$ which is prime.

Assume $p_1,\ p_2,\ \cdots,\ p_k$ are prime. Consider the number Q given by

$$Q = [p_1 \times p_2 \times \cdots \times p_k] + 1$$

The number Q is either prime or composite. If it is prime then we are done because we have found a prime $p_{k+1} = Q$ so by mathematical induction we have an infinite number of primes. If Q is composite then none of the primes in the list $p_1,\ p_2,\ \cdots,\ p_k$ goes into Q so there must be a prime different from $p_1,\ p_2,\ \cdots,\ p_k$ which we can call p_{k+1}. Hence again by mathematical induction we have an infinite number of primes.

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