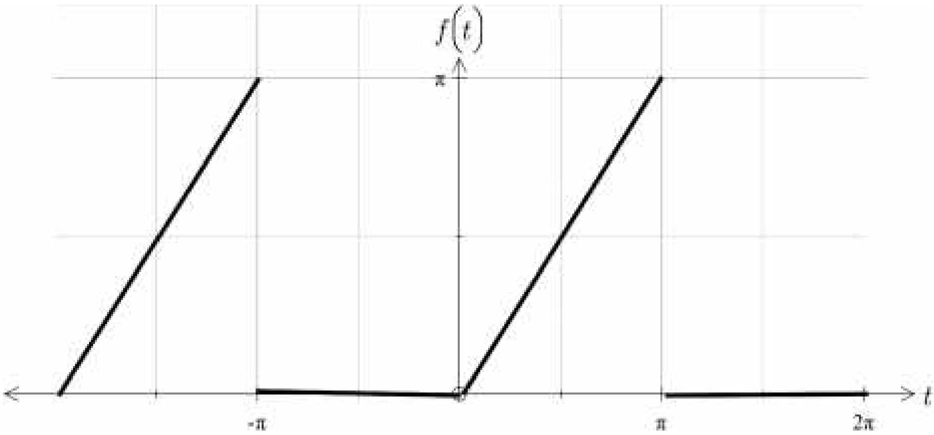
Complete Solutions to Exercises 17(c)

1. (a) (i) We are given the following graph:



This can be represented by the function $f(t) = \begin{cases} t & \text{if } 0 < t < \pi \\ 0 & \text{if } \pi < t < 2\pi \end{cases}$.

Using this to find the average value A_0 is given by

(17.3)

$$\begin{aligned} A_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t \\ &= \frac{1}{2\pi}\int_0^{\pi} (t)\,\mathrm{d}t && \left[\begin{array}{l}\text{Replacing } f(t) = t \text{ and} \\ \text{changing limits to 0 and } \pi\end{array}\right] \\ &= \frac{1}{2\pi}\left[\frac{t^2}{2}\right]_0^{\pi} && [\text{Integrating}] \\ &= \frac{1}{4\pi}\left[(\pi)^2 - 0^2\right] && \left[\begin{array}{l}\text{Substituting the limits} \\ \text{and taking out } 1/2\end{array}\right] \\ &= \frac{1}{4\not{\pi}}\left(\pi^{\not{2}}\right) = \frac{\pi}{4} && [\text{Cancelling}] \end{aligned}$$

We have the average of the given function is $A_0 = \frac{\pi}{4}$. *How do we find the cosine coefficients* A_k *?*

We use formula (17.4) and for $f(t)$ we substitute $f(t) = t$ and the limits 0 to π. This gives

$$\begin{aligned} A_k &= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[f(t)\cos(kt)\right]\mathrm{d}t \\ &= \frac{1}{\pi}\int_0^{\pi}\left[t\cos(kt)\right]\mathrm{d}t && (\dagger) \end{aligned}$$

How do we evaluate $\int_0^{\pi}\left[t\cos\left(kt\right)\right]\mathrm{d}t$ *?*

We need to use integration by parts formula. In our integral we have $u = t$ and $v' = \cos\left(kt\right)$:

$$u = t \qquad\qquad v' = \cos\left(kt\right)$$

$$u' = 1 \quad \left[\text{Differentiating}\right] \qquad\qquad v = \int \cos\left(kt\right)dt = \frac{\sin\left(kt\right)}{k}$$

Substituting these into integration by parts formula, $\int\left(uv'\right)\mathrm{d}t = uv - \int\left(u'v\right)\mathrm{d}t$, gives

$$\begin{aligned}
\int_0^{\pi}\left[t\cos\left(kt\right)\right]\mathrm{d}t &= \left[\frac{t\sin\left(kt\right)}{k}\right]_0^{\pi} - \int_0^{\pi}\left[\frac{\left(1\right)\sin\left(kt\right)}{k}\right]\mathrm{d}t && \left[\text{Using } \int\left(uv'\right)\mathrm{d}t = uv - \int\left(u'v\right)\mathrm{d}t\right]\\
&= \left[\frac{\pi\sin\left(\pi k\right)}{k} - 0\right] - \left[-\frac{\cos\left(kt\right)}{k^2}\right]_0^{\pi} && \left[\text{Integrating by } \int\sin\left(kt\right)\mathrm{d}t = -\frac{\cos\left(kt\right)}{k}\right]\\
&= \left[0-0\right] + \frac{1}{k^2}\left[\cos\left(\pi k\right) - \underbrace{\cos\left(0\right)}_{=1}\right] && \left[\text{Substituting limits and } \sin\left(\pi k\right) = 0\right]\\
&= \frac{1}{k^2}\left[\cos\left(\pi k\right) - 1\right]
\end{aligned}$$

Substituting this $\int_0^{\pi}\left[t\cos\left(kt\right)\right]\mathrm{d}t = \frac{1}{k^2}\left[\cos\left(\pi k\right) - 1\right]$ into (†) gives

$$A_k = \frac{1}{\pi}\int_0^{\pi}\left[t\cos\left(kt\right)\right]\mathrm{d}t = \frac{1}{k^2\pi}\left[\cos\left(\pi k\right) - 1\right] \qquad (*)$$

Recall that $\cos\left(k\pi\right) = \begin{cases} -1 & \text{if } k \text{ is odd} \\ 1 & \text{if } k \text{ is even}\end{cases}$.

If k is odd we have

$$A_k = \frac{1}{k^2\pi}\left[\cos\left(k\pi\right) - 1\right] = \frac{1}{k^2\pi}\left[-1-1\right] = -\frac{2}{k^2\pi}$$

If k is even we have

$$A_k = \frac{1}{k^2\pi}\left[\cos\left(k\pi\right) - 1\right] = \frac{1}{k^2\pi}\left[1-1\right] = 0$$

This implies there are *no* even cosine terms.

What else do we need to find?

The values of B_k. (The sine coefficients in the Fourier series).

$$B_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\left[f\left(t\right)\sin\left(kt\right)\right]\mathrm{d}t$$

$$= \frac{1}{\pi}\int_{0}^{\pi}\left[t\sin\left(kt\right)\right]\mathrm{d}t \qquad \left[\text{Remember } f\left(t\right)=t \text{ between 0 to } \pi\right]$$

$$B_k = \frac{1}{\pi}\int_{0}^{\pi}\left[t\sin\left(kt\right)\right]\mathrm{d}t \qquad (\dagger\dagger)$$

How do we evaluate this integral $\int_{0}^{\pi}\left[t\sin\left(kt\right)\right]\mathrm{d}t$ *?*

We need to use integration by parts again to find $\int_{0}^{\pi}\left[t\sin\left(kt\right)\right]\mathrm{d}t$:

$$u = t \qquad v' = \sin\left(kt\right)$$

$$u' = 1 \quad \left[\text{Differentiating}\right] \qquad v = \int \sin\left(kt\right)\mathrm{d}t = -\frac{\cos\left(kt\right)}{k}$$

Substituting this into integration by parts formula, $\int\left(uv'\right)\mathrm{d}x = uv - \int\left(u'v\right)\mathrm{d}x$, gives

$$\int_{0}^{\pi}\left[t\sin\left(kt\right)\right]\mathrm{d}t = -\left[\frac{t\cos\left(kt\right)}{k}\right]_{0}^{\pi} - \int_{0}^{\pi}\frac{\left(1\right)\left(-\cos\left(kt\right)\right)}{k}\mathrm{d}t$$

$$= -\frac{1}{k}\left[\pi\cos\left(k\pi\right)-0\right] + \frac{1}{k}\int_{0}^{\pi}\cos\left(kt\right)\mathrm{d}t \qquad \left[\begin{array}{l}\text{Substituting limits and}\\ \text{taking out common factor}\end{array}\right]$$

$$= -\frac{1}{k}\left[\pi\cos\left(k\pi\right)\right] + \frac{1}{k}\left[\frac{\sin\left(kt\right)}{k}\right]_{0}^{\pi} \qquad \left[\text{By } \int\cos\left(kt\right)\mathrm{d}t = \frac{\sin\left(kt\right)}{k}\right]$$

$$= -\frac{1}{k}\left[\pi\cos\left(k\pi\right)\right] + \frac{1}{k^2}\left[\underbrace{\sin\left(k\pi\right)}_{=0} - \underbrace{\sin\left(0\right)}_{=0}\right] \qquad \left[\text{Substituting limits}\right]$$

$$= -\frac{\pi}{k}\cos\left(k\pi\right) \qquad \left[\text{Simplifying}\right]$$

Putting this $\int_{0}^{\pi}\left[t\sin\left(kt\right)\right]\mathrm{d}t = -\frac{\pi}{k}\cos\left(k\pi\right)$ into $(\dagger\dagger)$ gives

$$B_k = \frac{1}{\pi}\int_{0}^{\pi}\left[t\sin\left(kt\right)\right]\mathrm{d}t = \frac{1}{\cancel{\pi}}\left[-\frac{\cancel{\pi}}{k}\cos\left(k\pi\right)\right] = -\frac{\cos\left(k\pi\right)}{k}$$

From above we have

$$\cos\left(k\pi\right) = \begin{cases} -1 & \text{if } k \text{ is odd} \\ 1 & \text{if } k \text{ is even}\end{cases}$$

Therefore if k is odd we have

$$B_k = -\frac{\cos\left(\pi k\right)}{k} = -\frac{\left(-1\right)}{k} = \frac{1}{k}$$

If k is even we have

$$B_k = -\frac{\cos(\pi k)}{k} = -\frac{(1)}{k} = -\frac{1}{k}$$

Putting these derived values,

$$A_0 = \frac{\pi}{4},\ A_k = \begin{cases} -2/k^2\pi & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases} \quad \text{and} \quad B_k = \begin{cases} 1/k & \text{if } k \text{ is odd} \\ -1/k & \text{if } k \text{ is even} \end{cases}$$

into the generic Fourier series

(17.2) $\quad f(t) = A_0 + A_1\cos(t) + A_2\cos(2t) + \ldots + B_1\sin(t) + B_2\sin(2t) + \ldots$

gives

$$f(t) = \frac{\pi}{4} - \frac{2}{\pi}\left(\cos(t) + \frac{\cos(3t)}{9} + \frac{\cos(5t)}{25} + \cdots\right) + \left(\sin(t) - \frac{\sin(2t)}{2} + \frac{\sin(3t)}{3} - \frac{\sin(4t)}{4} + \cdots\right)$$

(ii) Substituting $t = 0$ into the Fourier series of part (i) gives

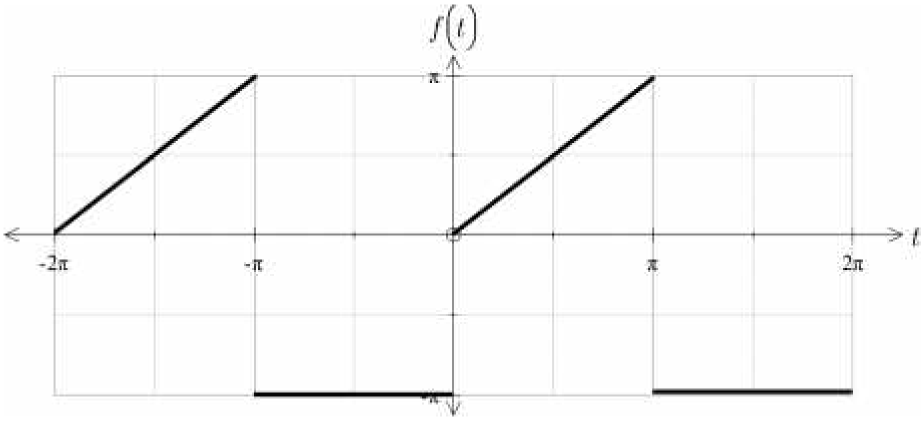
$$f(0) = \frac{\pi}{4} - \frac{2}{\pi}\left(\cos(0) + \frac{\cos(0)}{9} + \frac{\cos(0)}{25} + \frac{\cos(0)}{49} + \frac{\cos(0)}{81} + \cdots\right) + \underbrace{\left(\sin(0) - \frac{\sin(0)}{2} + \frac{\sin(0)}{3} - \cdots\right)}_{=0}$$

$$0 = \frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \cdots\right)$$

$$\frac{2}{\pi}\left(1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \cdots\right) = \frac{\pi}{4}$$

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \cdots = \frac{\pi^2}{8}$$

(b) The given graph is



This can be written as $f(t) = \begin{cases} t & \text{if } 0 < t < \pi \\ -\pi & \text{if } \pi < t < 2\pi \end{cases}$.

Using this to find the average value A_0 is given by

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t \tag{17.3}$$

$$= \frac{1}{2\pi}\left\{\int_0^{\pi}(t)\,\mathrm{d}t + \int_{-\pi}^{0}(-\pi)\,\mathrm{d}t\right\}$$

$$= \frac{1}{2\pi}\left\{\left[\frac{t^2}{2}\right]_0^{\pi} - \pi\left[t\right]_{-\pi}^{0}\right\} \qquad \left[\text{Integrating}\right]$$

$$= \frac{1}{2\pi}\left[\frac{\pi^2}{2} - \pi\left(0-(-\pi)\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{\pi^2}{2} - \pi^2\right]$$

$$= \frac{1}{2\pi}\left[-\frac{\pi^2}{2}\right] = -\frac{\pi}{4}$$

We have the average of the given function is $A_0 = -\dfrac{\pi}{4}$. *How do we find the cosine coefficients* A_k *?*

We use formula (17.4) and for $f(t)$ we substitute $f(t) = t$ and the limits 0 to π and $f(t) = -\pi$ between $-\pi$ and 0. This gives

$$A_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\left[f(t)\cos(kt)\right]\mathrm{d}t$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi}\left[t\cos(kt)\right]\mathrm{d}t + \int_{-\pi}^{0}(-\pi)\cos(kt)\,\mathrm{d}t\right\} \tag{†}$$

We have already evaluated the first integral on the right hand side of (†) in the previous part (a), it was $\int_0^{\pi}\left[t\cos(kt)\right]\mathrm{d}t = \dfrac{1}{k^2}\left[\cos(\pi k) - 1\right]$.

Evaluating he second integral on the right hand side

$$\int_{-\pi}^{0}(-\pi)\cos(kt)\,\mathrm{d}t = -\pi\left[\frac{\sin(kt)}{k}\right]_{-\pi}^{0}$$

$$= -\frac{\pi}{k}\left[\underbrace{\sin(0)}_{0} - \underbrace{\sin(-k\pi)}_{0}\right] = 0$$

Substituting these $\int_0^{\pi}\left[t\cos(kt)\right]\mathrm{d}t = \dfrac{1}{k^2}\left[\cos(\pi k) - 1\right]$ and $\int_{-\pi}^{0}(-\pi)\,\mathrm{d}t = 0$ into (†) gives

$$
\begin{aligned}
A_k &= \frac{1}{\pi}\left\{\int_0^{\pi}\left[t\cos\left(kt\right)\right]\mathrm{d}t + \int_{-\pi}^{0}\left(-\pi\right)\cos\left(kt\right)\mathrm{d}t\right\}\\
&= \frac{1}{\pi}\left\{\frac{1}{k^2}\left[\cos\left(\pi k\right)-1\right]-0\right\}\\
&= \frac{1}{k^2\pi}\left[\cos\left(\pi k\right)-1\right]
\end{aligned}
$$

Recall that $\cos\left(k\pi\right) = \begin{cases} -1 & \text{if } k \text{ is odd} \\ 1 & \text{if } k \text{ is even} \end{cases}$.

If k is odd we have

$$A_k = \frac{1}{\pi}\left\{\frac{1}{k^2}\left[\cos\left(k\pi\right)-1\right]\right\} = \frac{1}{\pi}\left\{\frac{1}{k^2}\left[\left(-1\right)-1\right]\right\} = -\frac{2}{k^2\pi}$$

If k is even we have

$$A_k = \frac{1}{\pi}\left\{\frac{1}{k^2}\left[\cos\left(\pi k\right)-1\right]\right\} = \frac{1}{\pi}\left\{\frac{1}{k^2}\left[1-1\right]\right\} = 0$$

This implies there are *no even* cosine terms.

What else do we need to find?

The values of B_k. (The sine coefficients in the Fourier series).

$$
\begin{aligned}
B_k &= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[f\left(t\right)\sin\left(kt\right)\right]\mathrm{d}t\\
&= \frac{1}{\pi}\left\{\int_0^{\pi}\left[t\sin\left(kt\right)\right]\mathrm{d}t + \int_{-\pi}^{0}\left(-\pi\right)\sin\left(kt\right)\mathrm{d}t\right\}\\
&= \frac{1}{\pi}\left\{\int_0^{\pi}\left[t\sin\left(kt\right)\right]\mathrm{d}t - \pi\int_{-\pi}^{0}\sin\left(kt\right)\mathrm{d}t\right\} \qquad (*)
\end{aligned}
$$

The first integral on the right we have already evaluated in part(a):

$$\int_0^{\pi}\left[t\sin\left(kt\right)\right]\mathrm{d}t = -\frac{\pi}{k}\cos\left(k\pi\right)$$

The second integral on the right is

$$
\begin{aligned}
\int_{-\pi}^{0}\sin\left(kt\right)\mathrm{d}t &= \left[-\frac{\cos\left(kt\right)}{k}\right]_{-\pi}^{0}\\
&= -\frac{1}{k}\left[\cos\left(0\right)-\cos\left(-k\pi\right)\right]\\
&= -\frac{1}{k}\left[1-\cos\left(k\pi\right)\right] \qquad \left[\text{because } \cos\left(-x\right)=\cos\left(x\right)\right]
\end{aligned}
$$

Substituting these results

$$\int_0^{\pi}\left[t\sin\left(kt\right)\right]\mathrm{d}t = -\frac{\pi}{k}\cos\left(\pi k\right) \text{ and } \int_{-\pi}^{0}\sin\left(kt\right)\mathrm{d}t = -\frac{1}{k}\left[1-\cos\left(k\pi\right)\right]$$

Into (*) gives

$$\begin{aligned} B_k &= \frac{1}{\pi}\left\{\int_0^\pi \left[t\sin\left(kt\right)\right]\mathrm{d}t - \pi\int_{-\pi}^0 \sin\left(kt\right)\,\mathrm{d}t\right\} \\ &= \frac{1}{\pi}\left\{-\frac{\pi}{k}\cos\left(\pi k\right) + \frac{\pi}{k}\left[1-\cos\left(k\pi\right)\right]\right\} \\ &= \frac{1}{\not{\pi}}\left(\frac{\not{\pi}}{k}\right)\left[-\cos\left(k\pi\right) + 1 - \cos\left(k\pi\right)\right] \qquad \left[\text{Taking out a factor of } \frac{\pi}{k}\right] \\ &= \frac{1}{k}\left[1 - 2\cos\left(k\pi\right)\right] \end{aligned}$$

From above we have

$$\cos\left(k\pi\right) = \begin{cases} -1 & \text{if } k \text{ is odd} \\ 1 & \text{if } k \text{ is even} \end{cases}$$

Therefore if k is odd we have

$$B_k = \frac{1}{k}\left[1-2\cos\left(k\pi\right)\right] = \frac{1}{k}\left[1-2\left(-1\right)\right] = \frac{3}{k}$$

If k is even we have

$$B_k = \frac{1}{k}\left[1-2\cos\left(k\pi\right)\right] = \frac{1}{k}\left[1-2\left(1\right)\right] = -\frac{1}{k}$$

Putting these derived values,

$$A_0 = -\frac{\pi}{4},\ A_k = \begin{cases} -2/k^2\pi & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases} \text{ and } B_k = \begin{cases} 3/k & \text{if } k \text{ is odd} \\ -1/k & \text{if } k \text{ is even} \end{cases}$$

into the generic Fourier series

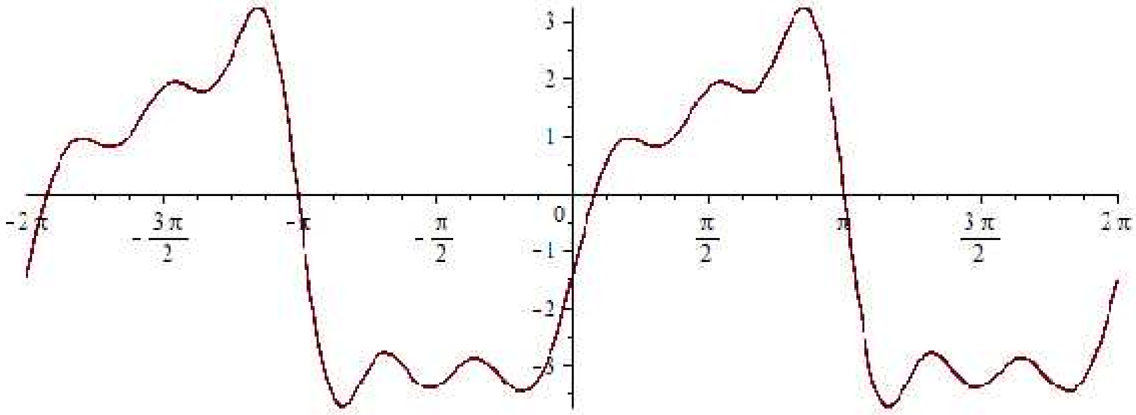
(17.2) $\quad f\left(t\right) = A_0 + A_1\cos\left(t\right) + A_2\cos\left(2t\right) + \ldots + B_1\sin\left(t\right) + B_2\sin\left(2t\right) + \ldots$

Gives

$$\begin{aligned} f\left(t\right) &= -\frac{\pi}{4} - \frac{2\cos\left(t\right)}{\pi} - \frac{2\cos\left(3t\right)}{9\pi} - \frac{2\cos\left(5t\right)}{25\pi} - \cdots \\ &\qquad + 3\sin\left(t\right) - \frac{\sin\left(2t\right)}{2} + \frac{3\sin\left(3t\right)}{3} - \frac{\sin\left(4t\right)}{4} + \frac{3\sin\left(5t\right)}{5} + \cdots \\ &= -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos\left(t\right) + \frac{\cos\left(3t\right)}{9} + \frac{\cos\left(5t\right)}{25} - \cdots\right) \\ &\qquad + 3\left(\sin\left(t\right) + \frac{\sin\left(3t\right)}{3} + \frac{\sin\left(5t\right)}{5} + \cdots\right) - \left(\frac{\sin\left(2t\right)}{2} + \frac{\sin\left(4t\right)}{4} + \frac{\sin\left(6t\right)}{6} + \cdots\right) \end{aligned}$$

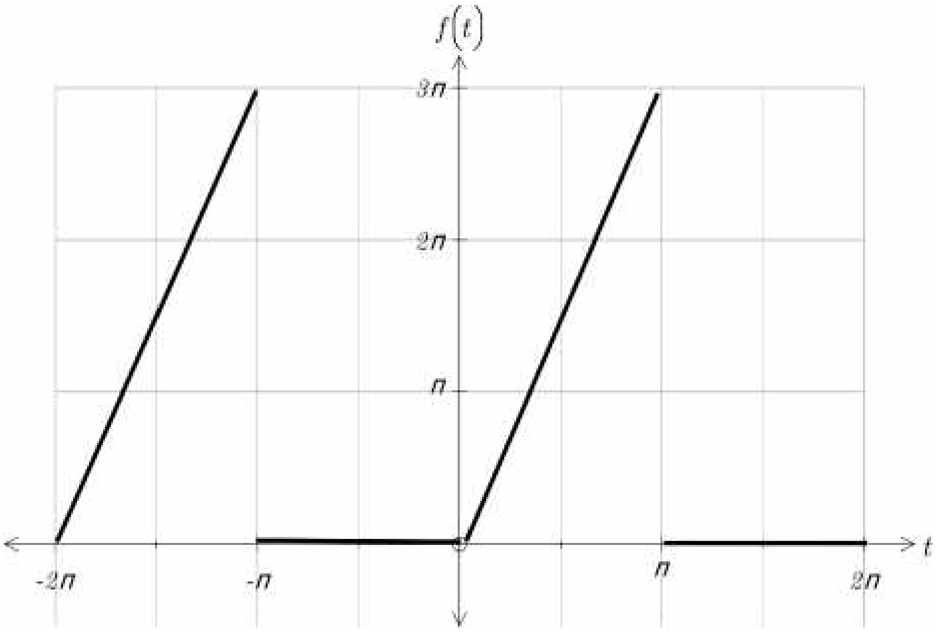
We can see the graphical visualization of this in Maple:

$$> -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos(t) + \frac{\cos(3t)}{9} + \frac{\cos(5t)}{25} + \frac{\cos(7t)}{49}\right) + 3\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5}\right) - \left(\frac{\sin(2t)}{2} + \frac{\sin(4t)}{4} + \frac{\sin(6t)}{6}\right)$$



2. (i) We need to sketch the given function:

$$f(t) = \begin{cases} 0 & -\pi < t < 0 \\ 3t & 0 < t < \pi \end{cases}$$



(ii) We need to find the Fourier series of this waveform.

We have already $f(t) = \begin{cases} 0 & -\pi < t < 0 \\ t & 0 < t < \pi \end{cases}$ in the previous question part (a).

Our given function is

$$3f(t) = \begin{cases} 0 & -\pi < t < 0 \\ 3t & 0 < t < \pi \end{cases}$$

We apply the following linearity property of section B:

Property 1:

If $f\left(t\right)$ has a Fourier series given by

$$f\left(t\right)=A_0+A_1\cos\left(t\right)+A_2\cos\left(2t\right)+\cdots+B_1\sin\left(t\right)+B_2\sin\left(2t\right)+\cdots$$

The Fourier series of $cf\left(t\right)$ where c is a constant is

$$cf\left(t\right)=cA_0+cA_1\cos\left(t\right)+cA_2\cos\left(2t\right)+\cdots+cB_1\sin\left(t\right)+cB_2\sin\left(2t\right)+\cdots$$

The answer to part (a) was

$$f\left(t\right)=\frac{\pi}{4}-\frac{2}{\pi}\left(\cos\left(t\right)+\frac{\cos\left(3t\right)}{9}+\frac{\cos\left(5t\right)}{25}+\cdots\right)+\left(\sin\left(t\right)-\frac{\sin\left(2t\right)}{2}+\frac{\sin\left(3t\right)}{3}-\frac{\sin\left(4t\right)}{4}+\cdots\right)$$

Therefore by the above linearity property we have

$$3f\left(t\right)=\frac{3\pi}{4}-\frac{6}{\pi}\left(\cos\left(t\right)+\frac{\cos\left(3t\right)}{9}+\frac{\cos\left(5t\right)}{25}+\cdots\right)+3\left(\sin\left(t\right)-\frac{\sin\left(2t\right)}{2}+\frac{\sin\left(3t\right)}{3}-\frac{\sin\left(4t\right)}{4}+\cdots\right)$$

3. (i) We are given $f\left(t\right)=\begin{cases}t^2 & \text{when } 0<t<2\pi\\ 2\pi^2 & \text{when } t=0 \text{ and } t=2\pi\end{cases}$ and it has a period of 2π. We need to find the Fourier series of this function.

The constant term A_0 is given by

$$A_0=\frac{1}{2\pi}\int_0^{2\pi}\left(t^2\right)\mathrm{d}t=\frac{1}{2\pi}\left[\frac{t^3}{3}\right]_0^{2\pi}$$
$$=\frac{1}{2\pi}\left[\frac{\left(2\pi\right)^3}{3}\right]=\frac{1}{2\not\pi}\left[\frac{8\pi^{\not 3}}{3}\right]=\frac{4\pi^2}{3}$$

The cosine coefficients A_k are given by

$$A_k=\frac{1}{\pi}\int_0^{2\pi}\left[t^2\cos\left(kt\right)\right]\mathrm{d}t \qquad (*)$$

How do we find $\int_0^{2\pi}\left[t^2\cos\left(kt\right)\right]\mathrm{d}t$ *?*

Use integration by parts formula; $\int uv'\,\mathrm{d}x=uv-\int\left(u'v\right)\mathrm{d}x$.

Let

$$u=t^2 \qquad v'=\cos\left(kt\right)$$
$$u'=2t \qquad v=\int\cos\left(kt\right)\mathrm{d}t=\frac{\sin\left(kt\right)}{k}$$

Applying the integration by parts formula gives

$$\begin{aligned}\int_0^{2\pi}\left[t^2\cos\left(kt\right)\right]\mathrm{d}t &= \frac{1}{k}\left[t^2\sin\left(kt\right)\right]_0^{2\pi} - \frac{1}{k}\int_0^{2\pi}\left[2t\sin\left(kt\right)\right]\mathrm{d}t \\ &= \frac{1}{k}\left[0\right] - \frac{2}{k}\int_0^{2\pi}\left[t\sin\left(kt\right)\right]\mathrm{d}t \\ &= -\frac{2}{k}\int_0^{2\pi}\left[t\sin\left(kt\right)\right]\mathrm{d}t \qquad (\dagger)\end{aligned}$$

We need to apply integration by parts again in order to find $\int_0^{2\pi}\left[t\sin\left(kt\right)\right]\mathrm{d}t$:

$$u = t \qquad v' = \sin\left(kt\right)$$

$$u' = 1 \qquad v = \int \sin\left(kt\right)\mathrm{d}t = -\frac{\cos\left(kt\right)}{k}$$

We have

$$\begin{aligned}\int_0^{2\pi}\left[t\sin\left(kt\right)\right]\mathrm{d}t &= -\frac{1}{k}\left[t\cos\left(kt\right)\right]_0^{2\pi} + \frac{1}{k}\int_0^{2\pi}\cos\left(kt\right)\mathrm{d}t \\ &= -\frac{1}{k}\left[2\pi\cos\left(2\pi k\right) - 0\right] + \frac{1}{k}\frac{1}{k}\left[\sin\left(kt\right)\right]_0^{2\pi} \\ &= -\frac{1}{k}\left[2\pi\right] + \frac{1}{k^2}\left[0\right] = -\frac{2\pi}{k}\end{aligned}$$

Substituting this $\int_0^{2\pi}\left[t\sin\left(kt\right)\right]\mathrm{d}t = -\frac{2\pi}{k}$ into (†) gives

$$\begin{aligned}\int_0^{2\pi}\left[t^2\cos\left(kt\right)\right]\mathrm{d}t &= -\frac{2}{k}\int_0^{2\pi}\left[t\sin\left(kt\right)\right]\mathrm{d}t \\ &= -\frac{2}{k}\left(-\frac{2\pi}{k}\right) = \frac{4\pi}{k^2}\end{aligned}$$

Putting this result $\int_0^{2\pi}\left[t^2\cos\left(kt\right)\right]\mathrm{d}t = \frac{4\pi}{k^2}$ into (*) yields

$$A_k = \frac{1}{\pi}\int_0^{2\pi}\left[t^2\cos\left(kt\right)\right]\mathrm{d}t = \frac{1}{\cancel{\pi}}\left(\frac{4\cancel{\pi}}{k^2}\right) = \frac{4}{k^2}$$

Similarly we evaluate the sine coefficients $B_k = \frac{1}{\pi}\int_0^{2\pi}\left[f\left(t\right)\sin\left(kt\right)\right]\mathrm{d}t$ with

$$B_k = \frac{1}{\pi}\int_0^{2\pi}\left[t^2\sin\left(kt\right)\right]\mathrm{d}t \qquad (**)$$

Because $f\left(t\right) = t^2$.

Applying the integration by parts formula with:

$$u = t^2 \qquad v' = \sin\left(kt\right)$$

$$u' = 2t \qquad v = \int \sin\left(kt\right)\,\mathrm{d}t = -\frac{\cos\left(kt\right)}{k}$$

We have

$$\begin{aligned}\int_0^{2\pi}\left[t^2\sin\left(kt\right)\right]\mathrm{d}t &= -\frac{1}{k}\left[t^2\cos\left(kt\right)\right]_0^{2\pi} + \frac{1}{k}\int_0^{2\pi}\left[2t\cos\left(kt\right)\right]\mathrm{d}t \\ &= -\frac{1}{k}\left[4\pi^2\underbrace{\cos\left(2\pi k\right)}_{=1} - 0\right] + \frac{2}{k}\int_0^{2\pi}\left[t\cos\left(kt\right)\right]\mathrm{d}t \\ &= -\frac{4\pi^2}{k} + \frac{2}{k}\int_0^{2\pi}\left[t\cos\left(kt\right)\right]\mathrm{d}t \qquad (\dagger\dagger)\end{aligned}$$

We need to apply integration by parts again in order to find $\int_0^{2\pi}\left[t\cos\left(kt\right)\right]\mathrm{d}t$:

$$u = t \qquad v' = \cos\left(kt\right)$$

$$u' = 1 \qquad v = \int \cos\left(kt\right)\,\mathrm{d}t = \frac{\sin\left(kt\right)}{k}$$

We have

$$\begin{aligned}\int_0^{2\pi}\left[t\cos\left(kt\right)\right]\mathrm{d}t &= \frac{1}{k}\left[t\sin\left(kt\right)\right]_0^{2\pi} - \frac{1}{k}\int_0^{2\pi}\sin\left(kt\right)\mathrm{d}t \\ &= \frac{1}{k}\left[0\right] - \frac{1}{k}\frac{1}{k}\left[-\cos\left(kt\right)\right]_0^{2\pi} \\ &= \frac{1}{k^2}\left[\underbrace{\cos\left(2\pi k\right)}_{=1} - \underbrace{\cos\left(0\right)}_{=1}\right] = 0\end{aligned}$$

Substituting this $\int_0^{2\pi}\left[t\cos\left(kt\right)\right]\mathrm{d}t = 0$ into $(\dagger\dagger)$ gives

$$\int_0^{2\pi}\left[t^2\sin\left(kt\right)\right]\mathrm{d}t = -\frac{4\pi^2}{k} + \underbrace{\frac{2}{k}\int_0^{2\pi}\left[t\cos\left(kt\right)\right]\mathrm{d}t}_{=0} = -\frac{4\pi^2}{k}$$

Putting this result $\int_0^{2\pi}\left[t^2\sin\left(kt\right)\right]\mathrm{d}t = -\frac{4\pi^2}{k}$ into (**) yields

$$B_k = \frac{1}{\pi}\int_0^{2\pi}\left[t^2\sin\left(kt\right)\right]\mathrm{d}t = \frac{1}{\cancel{\pi}}\left(-\frac{4\pi^{\cancel{2}}}{k}\right) = -\frac{4\pi}{k}$$

Substituting all the coefficients $A_0 = \dfrac{4\pi^2}{3}$, $A_k = \dfrac{4}{k^2}$ and $B_k = -\dfrac{4\pi}{k}$ into the Fourier series

(17.2) $\quad f(t) = A_0 + A_1\cos(t) + A_2\cos(2t) + \ldots + B_1\sin(t) + B_2\sin(2t) + \ldots$

Gives

$$f(t) = \frac{4\pi^2}{3} + 4\left[\cos(t) + \frac{\cos(2t)}{4} + \frac{\cos(3t)}{9} + \ldots\right] - 4\pi\left[\sin(t) + \frac{\sin(2t)}{2} + \frac{\sin(3t)}{3} + \ldots\right]$$

(ii) We need to show that $\zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots$. *How?*

By substituting $t = 0$ into the Fourier series of part (i). Since the given function is

$$f(t) = \begin{cases} t^2 & \text{when } 0 < t < 2\pi \\ 2\pi^2 & \text{when } t = 0 \text{ and } t = 2\pi \end{cases}$$

Therefore $f(0) = 2\pi^2$. We have

$$\underbrace{f(0)}_{=2\pi^2} = \frac{4\pi^2}{3} + 4\left[\cos(0) + \frac{\cos(0)}{4} + \frac{\cos(0)}{9} + \frac{\cos(0)}{16} + \frac{\cos(0)}{25} + \ldots\right] - 4\pi\underbrace{\left[\sin(0) + \frac{\sin(0)}{2} + \frac{\sin(0)}{3} + \ldots\right]}_{=0}$$

$$2\pi^2 = \frac{4\pi^2}{3} + 4\left[1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots\right]$$

$$\frac{2\pi^2}{3} = 4\left[1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots\right] \qquad \left[\text{Because } 2\pi^2 - \frac{4\pi^2}{3} = \frac{2\pi^2}{3}\right]$$

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots$$

Hence $\zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots$.

4. We need to find the Fourier series of $f(t) = t^2 + t,\ 0 < t < 2\pi$.

Let $g(t) = t^2,\ 0 < t < 2\pi$, $h(t) = t,\ 0 < t < 2\pi$ both with period 2π. Then by the result of the previous question we have the Fourier series of $g(t)$ is

$$g(t) = \frac{4\pi^2}{3} + 4\left[\cos(t) + \frac{\cos(2t)}{4} + \frac{\cos(3t)}{9} + \ldots\right] - 4\pi\left[\sin(t) + \frac{\sin(2t)}{2} + \frac{\sin(3t)}{3} + \ldots\right]$$

Also by Example 5 the Fourier series of $h(t)$ is

$$h(t) = \pi - 2\left[\sin(t) + \frac{\sin(2t)}{2} + \frac{\sin(3t)}{3} + \frac{\sin(4t)}{4} + \ldots\right]$$

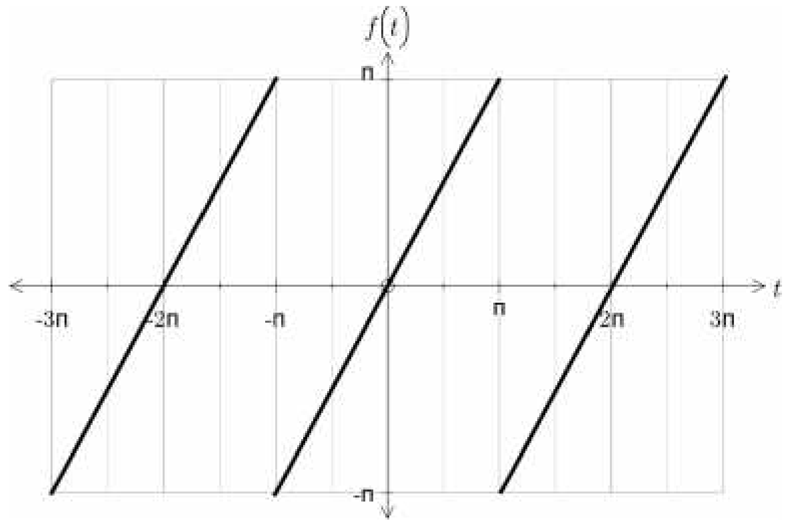
By the linearity property:

$$cf(t) + dg(t) = (cA_0 + dC_0) + (cA_1 + dC_1)\cos(t) + (cA_2 + dC_2)\cos(2t) + \cdots$$
$$+ (cB_1 + dD_1)\sin(t) + (cB_2 + dD_2)\sin(2t) + \cdots$$

We have

$$\begin{aligned}
f(t) &= g(t) + h(t)\\
&= \frac{4\pi^2}{3} + 4\left[\cos(t) + \frac{\cos(2t)}{4} + \frac{\cos(3t)}{9} + \ldots\right] - 4\pi\left[\sin(t) + \frac{\sin(2t)}{2} + \frac{\sin(3t)}{3} + \ldots\right]\\
&\quad + \pi - 2\left[\sin(t) + \frac{\sin(2t)}{2} + \frac{\sin(3t)}{3} + \frac{\sin(4t)}{4} + \ldots\right]\\
&= \left(\frac{4\pi^2}{3} + \pi\right) + 4\left[\cos(t) + \frac{\cos(2t)}{4} + \frac{\cos(3t)}{9} + \ldots\right]\\
&\quad - (4\pi + 2)\left[\sin(t) + \frac{\sin(2t)}{2} + \frac{\sin(3t)}{3} + \ldots\right]
\end{aligned}$$

5. (i) We need to sketch $f(t) = t, \ -\pi < t < \pi$ with period 2π:



(ii) The average value of this function over $-\pi$ to π is zero. Therefore $A_0 = 0$.

The cosine coefficients are given by

$$A_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,\mathrm{d}t = \frac{1}{\pi}\int_{-\pi}^{\pi}\left[t\cos(kt)\right]\mathrm{d}t \qquad (*)$$

We need to use integration by parts to evaluate the integral $\int_{-\pi}^{\pi}\left[t\cos(kt)\right]\mathrm{d}t$. Let

$$u = t \qquad v' = \cos(kt)$$

$$u' = 1 \qquad v = \int \cos(kt)\,\mathrm{d}t = \frac{1}{k}\sin(kt)$$

Applying the integration by parts formula; $\int (uv')\mathrm{d}x = uv - \int (u'v)\mathrm{d}x$ gives

$$\int_{-\pi}^{\pi} \left[t\cos(kt)\right]\mathrm{d}t = \frac{1}{k}\left[t\sin(kt)\right]_{-\pi}^{\pi} - \frac{1}{k}\int_{-\pi}^{\pi}\sin(kt)\,\mathrm{d}t \qquad \left[\text{By } \int (uv')\mathrm{d}x = uv - \int (u'v)\mathrm{d}x\right]$$

$$= \frac{1}{k}\left[\pi\underbrace{\sin(k\pi)}_{=0} - \left(-\pi\underbrace{\sin(-k\pi)}_{=0}\right)\right] - \frac{1}{k^2}\left[-\cos(kt)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{k^2}\left[\cos(k\pi) - \cos(-k\pi)\right]$$

$$= \frac{1}{k^2}\left[\cos(k\pi) - \cos(k\pi)\right] = 0 \qquad \left[\text{Because } \cos(-x) = \cos(x)\right]$$

Substituting this $\int_{-\pi}^{\pi}\left[t\cos(kt)\right]\mathrm{d}t = 0$ into (*) yields

$$A_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\left[t\cos(kt)\right]\mathrm{d}t = 0$$

This implies that there are *no* cosine terms.

Now we need to find the sine terms:

$$B_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\left[t\sin(kt)\right]\mathrm{d}t \qquad (**)$$

Again using integration by parts to find $\int_{-\pi}^{\pi}\left[t\sin(kt)\right]\mathrm{d}t$. Let

$$u = t \qquad v' = \sin(kt)$$

$$u' = 1 \qquad v = \int \sin(kt)\,\mathrm{d}t = -\frac{1}{k}\cos(kt)$$

Putting this into $\int (uv')\mathrm{d}x = uv - \int (u'v)\mathrm{d}x$ gives

$$\int_{-\pi}^{\pi}\left[t\sin(kt)\right]\mathrm{d}t = -\frac{1}{k}\left[t\cos(kt)\right]_{-\pi}^{\pi} + \frac{1}{k}\int_{-\pi}^{\pi}\cos(kt)\,\mathrm{d}t$$

$$= -\frac{1}{k}\left[\pi\cos(k\pi) - \left(-\pi\cos(-k\pi)\right)\right] + \frac{1}{k}\frac{1}{k}\left[\sin(kt)\right]_{-\pi}^{\pi}$$

$$= -\frac{1}{k}\left[\pi\cos(k\pi) + \pi\left(\underbrace{\cos(k\pi)}_{\text{Because } \cos(-x)=\cos(x)}\right)\right] + \frac{1}{k^2}\left[\underbrace{\sin(k\pi)}_{=0} - \underbrace{\sin(-k\pi)}_{=0}\right]$$

$$= -\frac{2\pi}{k}\cos(k\pi)$$

Substituting this $\int_{-\pi}^{\pi}\left[t\sin(kt)\right]\mathrm{d}t = -\frac{2\pi}{k}\cos(k\pi)$ into (**) yields

$$B_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\left[t\sin(kt)\right]\mathrm{d}t = \frac{1}{\cancel{\pi}}\left(-\frac{2\cancel{\pi}}{k}\cos(k\pi)\right) = -\frac{2}{k}\left[\cos(k\pi)\right]$$

Recall that

$$\cos(k\pi) = \begin{cases} 1 & \text{if } k = \text{even} \\ -1 & \text{if } k = \text{odd} \end{cases}$$

Our sine coefficients B_k are given by

$$B_k = -\frac{2}{k}\left[\cos(k\pi)\right] = \begin{cases} -2/k & \text{if } k = \text{even} \\ 2/k & \text{if } k = \text{odd} \end{cases}$$

Collecting all our Fourier coefficients; $A_0 = A_k = 0$ and

$$B_k = \begin{cases} -2/k & \text{if } k = \text{even} \\ 2/k & \text{if } k = \text{odd} \end{cases}$$

Putting these into the Fourier series yields

(17.2) $\quad f(t) = A_0 + A_1\cos(t) + A_2\cos(2t) + \ldots + B_1\sin(t) + B_2\sin(2t) + \ldots$

Gives

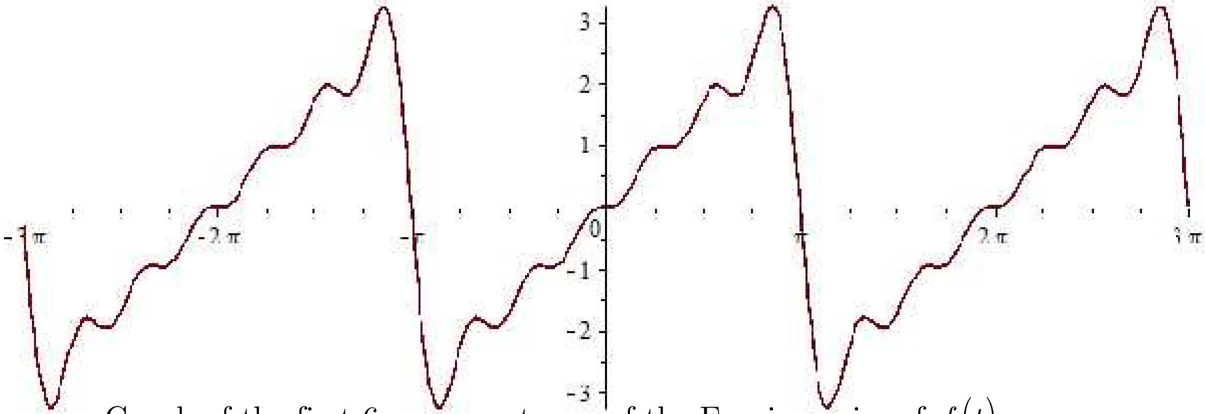
$$f(t) = 0 + \underbrace{0}_{\text{No cosine terms}} + 2\left[\sin(t) - \frac{\sin(2t)}{2} + \frac{\sin(3t)}{3} - \frac{\sin(4t)}{4} + \ldots\right]$$
$$= 2\left[\sin(t) - \frac{\sin(2t)}{2} + \frac{\sin(3t)}{3} - \frac{\sin(4t)}{4} + \ldots\right]$$

We can see the graphs of this Fourier series in Maple:

$$> f := t \to 2\left(\sin(t) - \frac{\sin(2\,t)}{2} + \frac{\sin(3\,t)}{3} - \frac{\sin(4\,t)}{4} + \frac{\sin(5\,t)}{5} - \frac{\sin(6\,t)}{6}\right)$$

$$f := t \to 2\sin(t) - \sin(2\,t) + \frac{2}{3}\sin(3\,t) - \frac{1}{2}\sin(4\,t) + \frac{2}{5}\sin(5\,t) - \frac{1}{3}\sin(6\,t)$$

> $plot(f, -3\pi..3\pi)$



Graph of the first 6 non-zero terms of the Fourier series of $f(t)$.

6. (i) We can write the given waveform as $f(t)=\begin{cases}0 & \text{when } -\pi < t < -\dfrac{\pi}{2}\\ 1 & \text{when } -\dfrac{\pi}{2} < t < \dfrac{\pi}{2}\\ 0 & \text{when } \dfrac{\pi}{2} < t < \pi\end{cases}$

The constant term or average value A_0 is given by

$$A_0 = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(1)\,\mathrm{d}t$$
$$= \frac{1}{2\pi}\Big[t\Big]_{-\pi/2}^{\pi/2}$$
$$= \frac{1}{2\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \frac{1}{2\pi}\left[\frac{\pi}{2}+\frac{\pi}{2}\right] = \frac{1}{2\cancel{\pi}}\left[\cancel{\pi}\right] = \frac{1}{2}$$

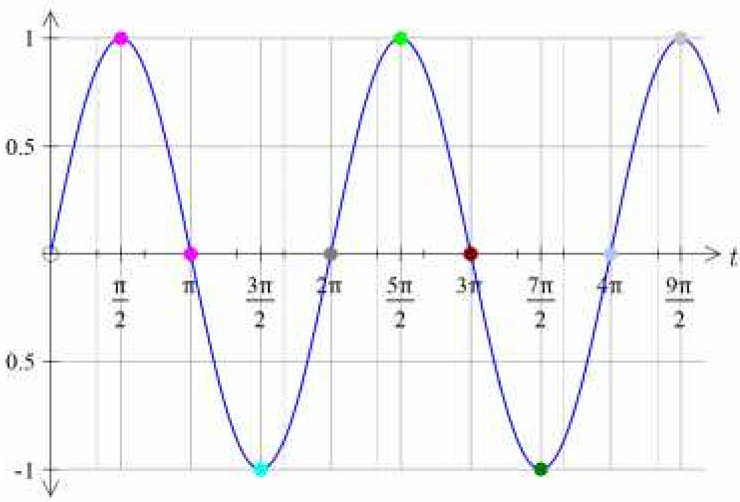
The cosine coefficients are given by

$$A_k = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(1)\cos(kt)\,\mathrm{d}t$$
$$= \frac{1}{k\pi}\Big[\sin(kt)\Big]_{-\pi/2}^{\pi/2}$$
$$= \frac{1}{k\pi}\left[\sin\left(k\frac{\pi}{2}\right) - \sin\left(-k\frac{\pi}{2}\right)\right]$$
$$= \frac{1}{k\pi}\left[\sin\left(k\frac{\pi}{2}\right) + \sin\left(k\frac{\pi}{2}\right)\right] \qquad \left[\text{Because } \sin(-x) = -\sin(x)\right]$$
$$= \frac{1}{k\pi}\left[2\sin\left(k\frac{\pi}{2}\right)\right]$$

The cosine coefficients are given by

$$A_k = \frac{2}{k\pi}\sin\left(k\frac{\pi}{2}\right) \qquad (*)$$

From the graph of the sine function:



We have

$$\sin\left(k\frac{\pi}{2}\right) = \begin{cases} 1 & \text{if } k = 1,\ 5,\ 9, \ldots \\ 0 & \text{if } k = \text{even} \\ -1 & \text{if } k = 3,\ 7,\ 11, \ldots \end{cases} \qquad (\ddagger)$$

Substituting this into (*) gives

$$A_k = \frac{2}{k\pi}\sin\left(k\frac{\pi}{2}\right) = \begin{cases} 2\,/\,k\pi & \text{if } k = 1,\ 5,\ 9, \ldots \\ 0 & \text{if } k = \text{even} \\ -2\,/\,k\pi & \text{if } k = 3,\ 7,\ 11, \ldots \end{cases} \qquad (**)$$

Similarly evaluating the sine coefficients gives

$$\begin{aligned} B_k &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} (1)\sin(kt)\,\mathrm{d}t \\ &= -\frac{1}{k\pi}\Big[\cos(kt)\Big]_{-\pi/2}^{\pi/2} \\ &= -\frac{1}{k\pi}\left[\cos\left(k\frac{\pi}{2}\right) - \cos\left(-k\frac{\pi}{2}\right)\right] \\ &= -\frac{1}{k\pi}\left[\cos\left(k\frac{\pi}{2}\right) - \cos\left(k\frac{\pi}{2}\right)\right] \qquad \Big[\text{Because } \cos(-x) = \cos(x)\Big] \\ &= -\frac{1}{k\pi}\big[0\big] = 0 \end{aligned}$$

This implies that there are *no* sine terms.

Substituting $A_0 = \dfrac{1}{2}$ and A_k is given by (**) into the generic Fourier series gives

(17.2) $f(t) = A_0 + A_1\cos(t) + A_2\cos(2t) + \ldots + B_1\sin(t) + B_2\sin(2t) + \ldots$

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\cos(t) + 0 - \frac{2}{3\pi}\cos(3t) + 0 + \frac{2}{5\pi}\cos(5t) + \cdots + \underbrace{0}_{\text{No sine terms}}$$
$$= \frac{1}{2} + \frac{2}{\pi}\left[\cos(t) - \frac{\cos(3t)}{3} + \frac{\cos(5t)}{5} - \right]$$

(ii) We need to deduce $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$.

We have to substitute $t = 0$ into our derived Fourier series of part (i). *What is* $f(0)$ *equal to?*

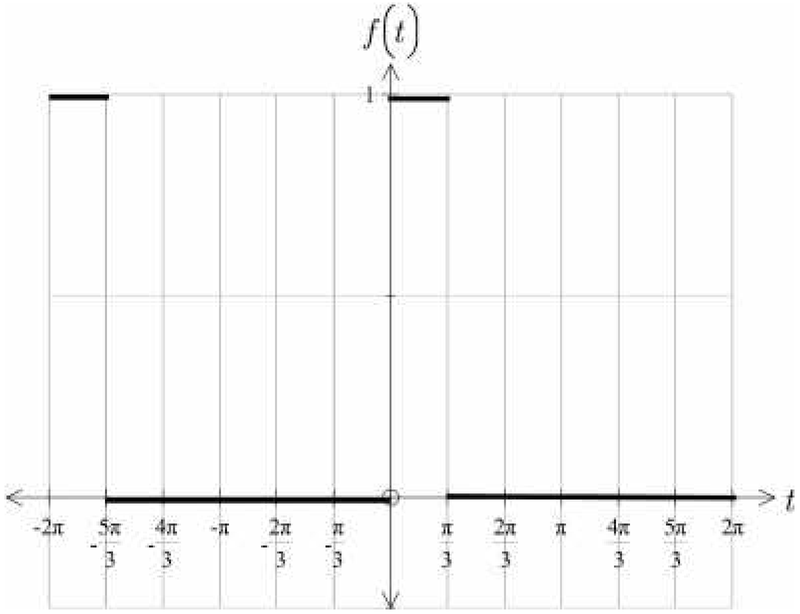
By looking at the given waveform we have $f(0) = 1$. Therefore

$$f(0) = \frac{1}{2} + \frac{2}{\pi}\left[\cos(0) - \frac{\cos(0)}{3} + \frac{\cos(0)}{5} - \frac{\cos(0)}{7} + \frac{\cos(0)}{9} - \cdots\right]$$
$$1 = \frac{1}{2} + \frac{2}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right]$$
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

7. (i) We need to sketch the following graph:

$$f(t) = \begin{cases} 0 & \text{when } -\pi < t < 0 \\ 1 & \text{when } 0 < t < \pi/3 \\ 0 & \text{when } \pi/3 < t < \pi \end{cases}$$

The graph of $f(t)$ between -2π and 2π:



(ii) The average value A_0 of this function is

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t = \frac{1}{2\pi}\int_{0}^{\pi/3} 1\,\mathrm{d}t = \frac{1}{2\pi}\big[t\big]_0^{\pi/3} = \frac{1}{2\not{\pi}}\left[\frac{\not{\pi}}{3}\right] = \frac{1}{6}$$

The cosine coefficients A_k are given by

$$\begin{aligned} A_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,\mathrm{d}t \\ &= \frac{1}{\pi}\int_{0}^{\pi/3} \cos(kt)\,\mathrm{d}t \\ &= \frac{1}{k\pi}\big[\sin(kt)\big]_0^{\pi/3} = \frac{1}{k\pi}\sin\left(k\frac{\pi}{3}\right) \end{aligned}$$

Substituting $k = 1, 2, 3, 4, 5, 6$ into $\sin\left(\frac{k\pi}{3}\right)$ gives

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2},\ \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2},\ \sin(\pi) = 0,$$

$$\sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2},\ \sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2} \text{ and } \sin(2\pi) = 0$$

For $k \geq 7$ these values are repeated as sine is a periodic function with period 2π.

Substituting each of these values into $A_k = \frac{1}{k\pi}\sin\left(k\frac{\pi}{3}\right)$ gives

$$A_1 = \frac{1}{\pi}\left(\frac{\sqrt{3}}{2}\right),\ A_2 = \frac{1}{2\pi}\left(\frac{\sqrt{3}}{2}\right),\ A_3 = 0,$$

$$A_4 = \frac{1}{4\pi}\left(-\frac{\sqrt{3}}{2}\right),\ A_5 = \frac{1}{5\pi}\left(-\frac{\sqrt{3}}{2}\right) \text{ and } A_6 = 0$$

Similarly finding the sine coefficients:

$$\begin{aligned} B_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,\mathrm{d}t \\ &= \frac{1}{\pi}\int_{0}^{\pi/3} \sin(kt)\,\mathrm{d}t \\ &= -\frac{1}{k\pi}\big[\cos(kt)\big]_0^{\pi/3} = -\frac{1}{k\pi}\left[\cos\left(\frac{k\pi}{3}\right) - 1\right] \end{aligned}$$

Substituting $k = 1, 2, 3, 4, 5, 6$ into $\cos\left(\frac{k\pi}{3}\right)$ gives

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2},\ \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2},\ \cos(\pi) = -1,$$

$$\cos\left(\frac{4\pi}{3}\right)=-\frac{1}{2},\ \cos\left(\frac{5\pi}{3}\right)=\frac{1}{2}\ \text{ and }\ \cos\left(2\pi\right)=1$$

Substituting each of these values into $B_k=-\frac{1}{k\pi}\left[\cos\left(\frac{k\pi}{3}\right)-1\right]$ gives

$$B_1=-\frac{1}{\pi}\left[\frac{1}{2}-1\right]=\frac{1}{2\pi},$$

$$B_2=-\frac{1}{2\pi}\left[-\frac{1}{2}-1\right]=-\frac{1}{2\pi}\left[-\frac{3}{2}\right]=\frac{3}{4\pi},$$

$$B_3=-\frac{1}{3\pi}\left[-1-1\right]=-\frac{1}{3\pi}\left[-2\right]=\frac{2}{3\pi},$$

$$B_4=-\frac{1}{4\pi}\left[-\frac{1}{2}-1\right]=-\frac{1}{4\pi}\left[-\frac{3}{2}\right]=\frac{3}{8\pi}$$

$$B_5=-\frac{1}{5\pi}\left[\frac{1}{2}-1\right]=-\frac{1}{5\pi}\left[-\frac{1}{2}\right]=\frac{1}{10\pi}$$

$$B_6=-\frac{1}{6\pi}\left[1-1\right]=0$$

Putting all these coefficients

$A_0=\frac{1}{6},\quad A_1=\frac{1}{\pi}\left(\frac{\sqrt{3}}{2}\right),\ A_2=\frac{1}{2\pi}\left(\frac{\sqrt{3}}{2}\right),\ A_3=0,\ A_4=\frac{1}{4\pi}\left(-\frac{\sqrt{3}}{2}\right),\ A_5=\frac{1}{5\pi}\left(-\frac{\sqrt{3}}{2}\right),$

$A_6=0,\ldots$ and the above B coefficients into

(17.2) $\quad f\left(t\right)=A_0+A_1\cos\left(t\right)+A_2\cos\left(2t\right)+\ldots+B_1\sin\left(t\right)+B_2\sin\left(2t\right)+\ldots$

Gives

$$\begin{aligned}f\left(t\right)&=\frac{1}{6}+\frac{\sqrt{3}}{2\pi}\left[\cos\left(t\right)+\frac{\cos\left(2t\right)}{2}+0-\frac{\cos\left(4t\right)}{4}-\frac{\cos\left(5t\right)}{5}+0+\cdots\right]\\&\quad+\frac{1}{\pi}\left[\frac{\sin\left(t\right)}{2}+\frac{3\sin\left(2t\right)}{4}+\frac{2\sin\left(3t\right)}{3}+\frac{3\sin\left(4t\right)}{8}+\frac{\sin\left(5t\right)}{10}+0+\cdots\right]\\&=\frac{1}{6}+\frac{\sqrt{3}}{2\pi}\left[\cos\left(t\right)+\frac{\cos\left(2t\right)}{2}-\frac{\cos\left(4t\right)}{4}-\frac{\cos\left(5t\right)}{5}+\cdots\right]\\&\quad+\frac{1}{\pi}\left[\frac{\sin\left(t\right)}{2}+\frac{3\sin\left(2t\right)}{4}+\frac{2\sin\left(3t\right)}{3}+\frac{3\sin\left(4t\right)}{8}+\frac{\sin\left(5t\right)}{10}+\cdots\right]\end{aligned}$$