Complete Solutions to Supplementary Problems on Mathematical Induction

Only available to tutors.

1. (i) Need to prove $6 \mid \left(2n^3+3n^2+n\right)$

Proof.

Clearly the result is true for $n=1$ because $6 \mid (2+3+1)$.

Assume the result is true for $n=k$:

$$6 \mid \left(2k^3+3k^2+k\right) \qquad (\dagger)$$

Required to prove

$$6 \mid \left(2(k+1)^3+3(k+1)^2+(k+1)\right) \qquad (\ddagger)$$

Expanding $2(k+1)^3+3(k+1)^2+(k+1)$ gives

$$\begin{aligned}2(k+1)^3+3(k+1)^2+(k+1) &= (k+1)\left[2(k+1)^2+3(k+1)+1\right] \qquad [\text{Factorizing}]\\ &= (k+1)\left[2\left(k^2+2k+1\right)+3k+3+1\right]\\ &= (k+1)\left[2k^2+4k+2+3k+4\right]\\ &= (k+1)\left[2k^2+7k+6\right]\\ &= 2k^3+7k^2+6k+2k^2+7k+6\\ &= 2k^3+9k^2+13k+6\\ &= \underbrace{2k^3+3k^2+k}_{=6m \text{ by } (\dagger)}+6k^2+12k+6\\ &= 6m+6\left(k^2+2k+1\right)\end{aligned}$$

Hence $2(k+1)^3+3(k+1)^2+(k+1)$ is a multiple of 6 or

$$6 \mid \left(2(k+1)^3+3(k+1)^2+(k+1)\right)$$

We have established (‡) so the proposition is true by mathematical induction.

■

(ii) Required to prove $12 \mid \left(n^4-n^2\right)$.

Proof.

Clearly we have $12 \mid \left(1^4-1^2\right)$ because $12\times 0=0$.

Assume the proposition is true for $n=k$:

$$12 \mid \left(k^4-k^2\right) \text{ or } k^4-k^2=12m \text{ where } m \text{ is an integer} \qquad (*)$$

We need to prove

$$12 \mid \left((k+1)^4-(k+1)^2\right) \qquad (**)$$

Manipulating $(k+1)^4-(k+1)^2$ we have

$$\begin{aligned}(k+1)^4-(k+1)^2&=(k+1)^2\left[(k+1)^2-1\right]\\&=(k+1)^2\left[k^2+2k\right]\\&=k(k+1)^2\left[k+2\right]\\&=k\left(k^2+2k+1\right)(k+2)\\&=k^4+4k^3+5k^2+2k\end{aligned}$$

Rewriting the last term so that we can use (*) gives

$$\begin{aligned}k^4+4k^3+5k^2+2k&=\underbrace{k^4-k^2}_{12m\text{ by }(*)}+4k^3+6k^2+2k\\&=12m+2\left(2k^3+3k^2+k\right)\end{aligned}$$

By result of part (i) we have $2k^3+3k^2+k=6j$ where *j* is an integer. Putting this into the above gives

$$k^4+4k^3+5k^2+2k=12m+2(6j)=12(m+j)$$

Therefore we have shown $12\left|\left((k+1)^4-(k+1)^2\right)\right.$ which means we have proven the proposition by mathematical induction.

■

2. We need to prove Fermat's Little Theorem which says $p\mid n^p-n$ where *p* is prime.

Proof.

We use mathematical induction. Clearly the result is true for $n=1$ because $1^p-1=0$ and $p\times 0=0$ so we have

$$p\mid 1^p-1$$

Assume the result holds for $n=k$ where *k* is a natural number:

$$p\mid k^p-k \text{ or } pm=k^p-k \text{ for some integer } m \qquad (*)$$

Required to prove

$$p\left|\left[(k+1)^p-(k+1)\right]\right.$$

Expanding the first term on RHS of this divisor by using the binomial theorem gives

$$(k+1)^p=k^p+pk^{p-1}+\frac{p(p-1)}{2!}k^{p-2}+\frac{p(p-1)(p-2)}{3!}k^{p-3}+\cdots+pk+1$$

Subtracting $k+1$ from this gives

$$\begin{aligned}(k+1)^p-(k+1)&=k^p+pk^{p-1}+\frac{p(p-1)}{2!}k^{p-2}+\frac{p(p-1)(p-2)}{3!}k^{p-3}+\cdots+pk+1-k-1\\&=\underbrace{k^p-k}_{=mp\text{ by }(*)}+pk^{p-1}+\frac{p(p-1)}{2!}k^{p-2}+\frac{p(p-1)(p-2)}{3!}k^{p-3}+\cdots+pk\\&=mp+p\underbrace{\left[k^{p-1}+\frac{(p-1)}{2!}k^{p-2}+\frac{(p-1)(p-2)}{3!}k^{p-3}+\cdots+k\right]}_{\text{Required to prove this is an integer}}\end{aligned}$$

$$\frac{p!}{r!(p-r)!} = \frac{p(p-1)(p-2)\cdots(p-r+1)(p-r)!}{r!(p-r)!}$$

$$= \frac{p(p-1)(p-2)\cdots(p-r+1)}{r!} \quad \left[\text{Cancelling } (p-r)!\right]$$