$\cancel{|x|^2 = x^2}$ $|x^2| = x^2$

$$y = \begin{cases} y & \text{if } y \geq 0 \\ -y & \text{if } y < 0 \end{cases}$$

Proof. Since $x^2 \geq 0$ (1)

$$|x^2| = x^2$$

(b) $|x|^2 = x^2$

If $x \geq 0$ then

$$|x|^2 = |x|\,|x| = x\,x = x^2$$

If $x < 0$ then

$$|x|^2 = |x|\,|x| = (-x)(-x) = x^2$$

Let $a \in \mathbb{R}$ & $a \geq 0$ then $|x| \leq a \iff -a \leq x \leq a$.

Proof: ($\Rightarrow$). We have $|x| \leq a$. If $x \geq 0$ then

$$|x| = x \leq a$$

If $x < 0$ then $|x| = \cancel{a}\ \underline{-x \leq a}$

x bis by (-1)

$$x \geq -a$$

$$\underline{-a \leq x \leq a.}$$

($\Leftarrow$) Assume $-a \leq x \leq a \Rightarrow |x| \leq a.$

$$|x| \le a \iff -a \le x \le a$$

$$|x| < a \iff -a < x < a$$

$\forall x \in \mathbb{R}$ we have $-|x| \le x \le |x|$

Proof: Substituting $a = |x|$ into the above gives

$$-|x| \le x \le |x|$$

Prop (L.17): $\forall x, y \in \mathbb{R}$ we have

$$|xy| = |x|\,|y|$$

Proof: If $x \geq 0$, $y \geq 0$ then

$$|x|\,|y| = xy \geq 0$$
$$= |xy|$$

If $x \geq 0$ but $y < 0$: $xy < 0$

$$|x|\,|y| = x(-y)$$
$$= -(xy)$$
$$= |xy|$$

$$|-x| = |x| \qquad |xy| = |x|\,|y|$$

Pf: $|-x| = |(-1) \times x| = |-1|\,|x| = |x|$ $\square$

$$|x+y| \leq |x| + |y|.$$

Proof:

$$\cancel{0 \leq (x+y)^2}$$

$$\begin{aligned}
0 \leq |x+y|^2 &= (x+y)^2 \\
&= x^2 + 2xy + y^2 \\
&= |x|^2 + 2xy + |y|^2
\end{aligned}$$

$$x \leq |x|$$
$$y \leq |y|$$

$$\begin{aligned}
&\leq |x|^2 + 2|x||y| + |y|^2 \\
&= (|x| + |y|)^2
\end{aligned}$$

$$|x+y|^2 \leq (|x| + |y|)^2$$

$$|x+y| \leq |x| + |y|$$

$\square$