Inequalities

$$a > b \iff a - b > 0$$

$$a > 3 \iff a - 3 > 0$$

$$a < 3 \iff a - 3 < 0$$

$$a > b \iff b < a$$

$$(a > b) \equiv (b < a)$$

$$(5 > 2) \equiv (2 < 5)$$

$$(\pi < 4) \equiv 4 > \pi$$

If $a > 0$ & $b > 0$ then $a + b > 0$

$a > 0$ & $b > 0$ then $ab > 0$

If $a \in \mathbb{R}$ then only one of the following is true:

$$a > 0, \; a = 0, \; \text{or} \; -a > 0$$

(i) If $a > b$ then

$$a + c > b + c \qquad c \in \mathbb{R}$$

Proof: From $a > b$ then $a - b > 0$. Therefore

$$a - b = a + c - c - b$$

$$= a + c - (b + c) > 0$$

$$a + c > b + c$$

$\square$

(ii) If $a>b$ & $b>c$ then $a>c$. ②

Proof: $a-b>0$ & $b-c>0$

$$a\underbrace{-b+b}_{=0}-c>0$$

$$a-c>0$$

$$a>c$$

□

If $a>b$ & $c>0$ then $ac>bc$.

Proof: From ~~a-b~~ $a>b$ we have

$$a-b>0$$

$$(a-b)c>0$$

$$ac-bc>0$$

$$ac>bc$$

□

$\boxed{\text{If } a>b \text{ \& } c<0 \text{ then } ac<bc}$

Proof: Since $c<0$ so $-c>0$. From $a>b$ we have

$$a-b>0$$

$$-c(a-b)>0$$

$$-ca+cb>0$$

$$bc>ac \Rightarrow ac<bc.$$

□

Prop (a) $1>0$

(b) Let $n \in \mathbb{N}$ then $n>0$.

Proof: $1 \neq 0$. Then let $x=1$ & we have

$$1 = 1^2 > 0.$$

Proof of (b): $1>0$ ✓

Assume the result is true for $n=k$.

$$k>0. \quad (*)$$

R.t.p the result for $k+1>0$.

$$\begin{array}{c} k>0 \\ 1>0 \\ \hline k+1>0 \end{array}$$

There is no smallest positive number.

Proof: Proof by contradiction. Suppose x is the smallest positive real number.

$$x>0.$$

Consider the number $\frac{1}{2}x$.

$$\frac{1}{2}<1$$

$$0<\frac{1}{2}x<x$$

Let $x \in \mathbb{R}$ & ~~$\forall \varepsilon > 0$~~ for every $\varepsilon > 0$ we have

$0 \le \underline{x < \varepsilon}$. Then $x = 0$.

Proof: Suppose $x \neq 0$. Then $x > 0$. (Le)

$$x > \frac{1}{2}x > 0.$$

Let $\varepsilon = \frac{1}{2}x$.

$$0 \le x < \frac{1}{2}x.$$

$$\varepsilon = 0.000\,000000 - -01$$

If $ab > 0$ then either

(i) a & b are positive or

(ii) both a & b are negative.

Proof: WLOG let $a > 0$ & $b < 0$.

$$a > 0$$

$$ab < 0$$

$x > 0$
$y < 0$
$xy < 0$

■