SECTION A **Inequalities**

By the end of this section you will be able to

- understand and use the symbols of inequality
- establish basic rules of inequalities
- derive properties of inequalities

This is a challenging section because we treat inequalities in an abstract manner and many students have difficulty in relating to the results obtained in this section. Additionally, students apply rules of inequalities as they would for equality, =, because they are comfortable in using them for equality. You need to be a lot more careful with inequalities and **cannot** blindly apply the same rules as you did for equality. Make sure you follow each step carefully and don't be confused with the notation involved with inequalities.

A1 Examples of Inequalities

In this section we discuss inequality of real numbers. *How do we represent real numbers graphically?*

By the real number line:

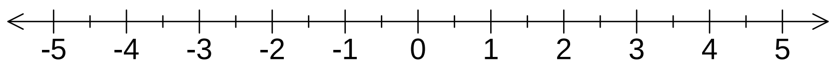


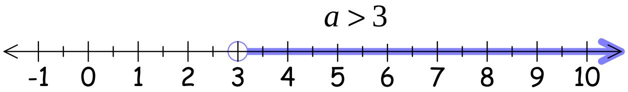
Fig 1

What does the term 'inequality' mean in the general sense?

We often read in the papers 'inequality of wealth' which generally implies that the distribution of wealth has **not** been equally dispersed.

Generally inequality is opposite to equality which implies that inequality means **not** equal to. For example we can say 'a is greater than 3' and this is written as '$a > 3$' meaning that the real number a is to the right of 3 on the real number line:

Fig 2



The real number a is less than 3 is denoted by $a < 3$ and means that a is to the left of 3 on the real number lir $a < 3$

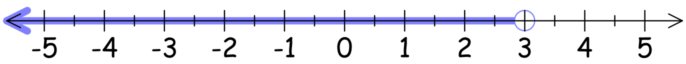


Fig 3

The notation $a \geq 3$ means that $a > 3$ or $a = 3$ and is drawn as:

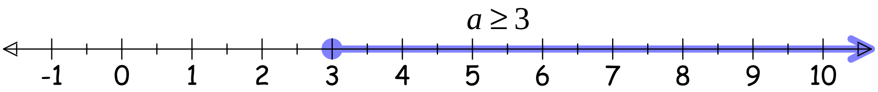


Fig 4

A2 Inequality Properties of Real Numbers

Let a and b be real numbers, $a \in \mathbb{R}$ and $b \in \mathbb{R}$, then **only** one of the following is true:

(i) $a > b$ (a is greater than b).

(ii) $a = b$ (a is equal to b).

(iii) $a < b$ (a is less than b).

The real numbers a and b satisfying $a > b \Leftrightarrow a - b > 0$. For example in the above case $a > 3 \Leftrightarrow a - 3 > 0$ and $a < 3 \Leftrightarrow a - 3 < 0$.

Also $a > b$ (a is greater than b) is equivalent to $b < a$ (b is less than a).

This can be written in mathematical notation as:

$$(a > b) \equiv (b < a)$$

For example $5 > 3$ (5 is greater than 3) is equivalent to $3 < 5$ (3 is less than 5).

What is ® < 4 *equivalent to?*

$4 >$ ®

We assume the following properties of inequalities of real numbers:

(4.1) If $a > 0$ and $b > 0$ then $a + b > 0$. *What does this notation mean?*

Adding two positive real numbers gives a positive real number.

(4.2) If $a > 0$ and $b > 0$ then $ab > 0$. *What does this notation mean?*

Multiplying two positive real numbers gives a positive real number.

(4.3) If $a \in \mathbb{R}$ then **only** one of the following is true: $a > 0$, $a = 0$ or $-a > 0$.

From these we prove other properties of inequalities of real numbers.

Proposition (4.4). Let a, b and c be real numbers. Then the following properties hold:

(a) If $a > b$ then $a + c > b + c$

(b) If $a > b$ and $b > c$ then $a > c$

Proof.

(a) Remember $a > b$ means $a - b > 0$. Therefore

$$a - b = a + \underbrace{c - c}_{=0} - b \qquad [\text{Rewriting} \quad a - b]$$
$$= a + c - (b + c) > 0 \qquad [\text{Because} \quad a - b > 0]$$

From $a + c - (b + c) > 0$ we have $a + c > b + c$. We have proven $a > b \Rightarrow a + c > b + c$.

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(b) From the given two inequalities, $a > b$ and $b > c$, we have

$$a - b > 0 \text{ and } b - c > 0$$

Adding these two positive inequalities together gives

$$a \underbrace{-b + b}_{=0} - c > 0$$
$$a - c > 0$$

From $a - c > 0$ we have the required result, $a > c$.

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We can apply the above proposition (4.4) to particular examples:

We know $3 > 2$. Adding $c = 5$ does **not** change the inequality:

$$3 + 5 > 2 + 5$$

Also adding $c = -5$ does **not** change the inequality:

$$3 + (-5) > 2 + (-5)$$

Examples of proposition (4.4) (b) ($a > b$ and $b > c \Rightarrow a > c$) are

$$3 > 2 \text{ and } 2 > 1 \Rightarrow 3 > 1$$
$$-3 > -4 \text{ and } -4 > -5 \Rightarrow -3 > -5$$

Proposition (4.5). Let a, b and c be real numbers. Then the following properties hold:

(a) If $a > b$ and $c > 0$ then $ac > bc$

(b) If $a > b$ and $c < 0$ then $ac < bc$ [The inequality changes].

Proof.

(a) Remember $a > b$ means that $a - b > 0$. We are given that $c > 0$ and so multiplying these two positive inequalities we have

$$(a - b)c > 0 \qquad [\text{By (4.2)}]$$
$$ac - bc > 0 \qquad [\text{Expanding}]$$
$$ac > bc$$

Hence we have our result.

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(b) If $c<0$ then $-c>0$ and we are given that $a>b$ so $a-b>0$.
Multiplying these two positive inequalities, $a-b>0$ and $-c>0$, we have

$$(a-b)(-c)>0 \qquad [\text{By } (4.2)]$$
$$-ac+bc>0 \qquad [\text{Expanding}]$$
$$bc>ac$$

Note that $bc>ac$ (bc is greater than ac) is the same as $ac<bc$ (ac is less than bc) which is the required result.

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Notice the result of the above proposition (4.5) (b). If we multiply the inequality $a>b$ by a negative real number such as $c<0$ then it changes the inequality sign to $ac<bc$. A negative multiple changes the inequality sign.
If $a<b$ and $c<0$ then what is the inequality between ac and bc?

$$ac>bc$$

Remember a negative multiple changes the inequality sign. Note that inequalities such as $<$ and $>$ do **not** behave like equality, $=$. You need to be a lot more careful with inequalities as the following application of proposition (4.5) shows.

$5>2$ and $c=3 \Rightarrow (5\times 3)>(2\times 3)$ [Inequality remains the same]
$5>2$ and $c=-3 \Rightarrow (5\times -3)<(2\times -3)$ [Inequality Changes]
$-2<-1$ and $c=-4 \Rightarrow (-2\times(-4))>(-1\times(-4))$ [Inequality Changes]

Proposition (4.6). Let x be a non-zero real number. Then $x^2>0$.

Comment. Since x is a non-zero real number therefore $x\neq 0$. *What can x be?* Because the real number x is **not zero** therefore it is either positive or negative which is denoted as $x>0$ or $x<0$.
Proof.
Consider the positive case first. Let $x>0$ then

$$xx>0 \qquad [\text{By } (4.2)]$$
$$x^2>0$$

Now consider the negative case. Let $x<0$ [Negative] then multiplying this by the same negative real number, $x<0$, we change the inequality:

$$xx > 0$$
$$x^2 > 0$$

We have proven our proposition that if x is a non zero real number then $x^2 > 0$.

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Proposition (4.7).

(a) $1 > 0$
(b) Let $n \in \mathbb{N}$ then $n > 0$

Proof.

(a) $1 = 1^2 > 0$ [With $x = 1$ in the above Proposition (4.6)]

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(b) *What does* $n \in \mathbb{N}$ *mean?*

$n \in \mathbb{N}$ means that n is a natural number, 1, 2, 3, 4, … *How do we prove* $n > 0$ *for* $n \in \mathbb{N}$ *?*

By using induction because it is a result concerning the natural numbers. The procedure for induction outlined in chapter 1 is to show the result for $n = 1$, assume it is true for $n = k$ and then prove the given result for $n = k + 1$.

By (a) we have $1 > 0$. Assume $k > 0$. Required to prove $k + 1 > 0$. We have

$$k > 0 \quad \text{[By Assumption]}$$
$$1 > 0 \quad \text{[By (a)]}$$
$$k + 1 > 0$$

Hence by induction we have proven our result, if $n \in \mathbb{N}$ then $n > 0$.

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Proposition (4.8)

There is **no** smallest positive real number.

How can we prove the given proposition?

By contradiction.

Proof.

Suppose there is a smallest positive real number, call it $x > 0$.

We have $0 < \frac{1}{2} < 1$, *why?*

Because

$$\frac{1}{2} < 1$$

$$\frac{1}{2} - \frac{1}{2} < 1 - \frac{1}{2} = \frac{1}{2} \qquad \left[\text{Subtract } \frac{1}{2} \text{ from both sides}\right]$$

$$0 < \frac{1}{2}$$

We have the inequality $0 < \frac{1}{2} < 1$ and multiplying this by a positive real number, $x > 0$, gives

$$0 < \frac{1}{2}x < x$$

Hence the positive real number $\frac{1}{2}x$ is less than the smallest positive real number, x. Contradiction! There is **no** smallest positive real number.

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Proposition (4.9)

Let x be a real number such that for every real $\pounds > 0$ we have $0 \le x < \pounds$, then $x = 0$.

Comment. $\pounds$ is the Greek letter epsilon. This Greek letter is used extensively in mathematics particularly in mathematical analysis so you should familiarize yourself with this Greek letter, $\pounds$. Don't be put off by this symbol. Many students do find the epsilon symbol, $\pounds$, off putting because they have not seen it before and so are uncomfortable in using it.

What does every real $\pounds > 0$ *mean?*

Every $\pounds > 0$ means every positive real number.

What does the given proposition 'for every real $\pounds > 0$ *we have* $0 \le x < \pounds$, *then* $x = 0$ *' mean?*

Means that if a real number such as $x \ge 0$ and it is less than every positive real number then $x = 0$. In everyday language it means that if x can be made as close as we please to zero then $x = 0$. *How can we prove the given proposition?*

By contradiction.

Proof.

Suppose $x \ne 0$. Then $x > 0$. *Why?*

Because we are given $0 \le x < £$ and if $x \ne 0$ then $x > 0$. We have the inequality

$$x > \frac{1}{2}x > 0$$

Since $\frac{1}{2}x$ is a positive real number so let $£ = \frac{1}{2}x$ because $£$ is a positive real number.

We have $x > \frac{1}{2}x = £$ that is $x > £$, but we are given $x < £$. Contradiction!

How?

Because we have x is greater than $£$ (epsilon) and x is less than $£$ (epsilon).

Since we have a contradiction therefore our supposition, $x \ne 0$, must be wrong.

Hence $x = 0$.

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Proposition (4.10)

If $ab > 0$ $[\text{Positive}]$ then either

(i) both a and b are positive or

(ii) both a and b are negative

Comment. We can use proof by contradiction again. *How do we apply proof by contradiction?*

Assume the result is false and then using this derive a contradiction by logical deductions. *What is our false assumption?*

We can say without loss of generality, assume a is positive and b is negative. *What does 'without loss of generality' mean?*

Without loss of generality is a simplifying assumption that we discussed in chapter 1. Instead of covering all the cases we assume one case and prove it for that case. In this example we assume a is positive and b is negative. We could have covered both cases:

1. a is positive and b is negative
2. a is negative and b is positive

It makes the proof longer and nothing is gained by inclusion of both cases.

Proof.

Since $ab > 0$ therefore $a \neq 0$ and $b \neq 0$. *Why not?*

Because if either of a or b is zero then $ab = 0$.

Without loss of generality assume a is positive and b is negative. This means in symbolic form we have

$$a > 0 \ [\text{Positive}] \text{ and } \quad b < 0 \ [\text{Negative}]$$

Multiplying $a > 0$ by $b < 0$ [Negative] gives

$$ab < 0 \quad [\text{Negative}]$$

Why?

Because by (4.5) (b) we know that when we multiply by a negative quantity, $b < 0$, then the inequality sign changes. Hence $a > 0$ changes to $ab < 0$ because $b < 0$.

The deduction $ab < 0$ is a contradiction! *Why?*

Because initially we are given $ab > 0$ [Positive] and now by our deduction we have $ab < 0$ [Negative].

Hence our assumption a is positive and b is negative must be false. Clearly a is negative and b is positive will give the same result so therefore both a and b have the same polarity. That is a and b are both positive or both negative. Hence our result.

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SUMMARY

In this section we have established properties of inequalities.

Let a, b and c be real numbers then we have the following properties:

(4.5) (a) If $a > b$ and $c > 0$ then $ac > bc$

(b) If $a > b$ and $c < 0$ then $ac < bc$

(4.9) If for every positive £ we have $0 \leq x < £$ then $x = 0$.

(4.10) If $ab > 0$ then both a and b have the same polarity.