SECTION E **Modulus Function**

By the end of this section you will be able to

- visualize the geometric interpretation of the modulus function
- derive some inequalities involving the modulus function

The first part of this section is straightforward. However for later parts you will need to know your work on inequalities to derive properties of the modulus function.

E1 Introduction to the Modulus Function

Definition (4.11). Let x be a real number then the **modulus** of x is denoted by $|x|$ and is defined as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

What does this mean?

If x is **positive or zero** then $|x| = x$ and if x is **negative** then $|x| = -x$. *What is the value of* $|5|$ *?*

By definition (4.11) we have $|5| = 5$. *What is the value of* $|-5|$ *?*

By definition (4.11) we have $|-5| = -(-5) = 5$. *What is the value of* $\left|-\frac{1}{2}\right|$ *?*

Similarly $\left|-\frac{1}{2}\right| = -\left(-\frac{1}{2}\right) = \frac{1}{2}$. *What is the value of* $\left|\frac{1}{3}\right|$ *?*

$\left|\frac{1}{3}\right| = \frac{1}{3}$. *What is the value of* |−®| *?*

|−®| = −(−®) = ® .

What do you notice about your results?

They are all positive. *What is* $|0|$ *equal to?*

$$|0| = 0$$

What does the graph of $|x|$ *look like?*

Note that the modulus of negative values of x, $|-x|$, is positive x and for positive and zero x we have $|x| = x$. Therefore the graph of $y = |x|$ is:

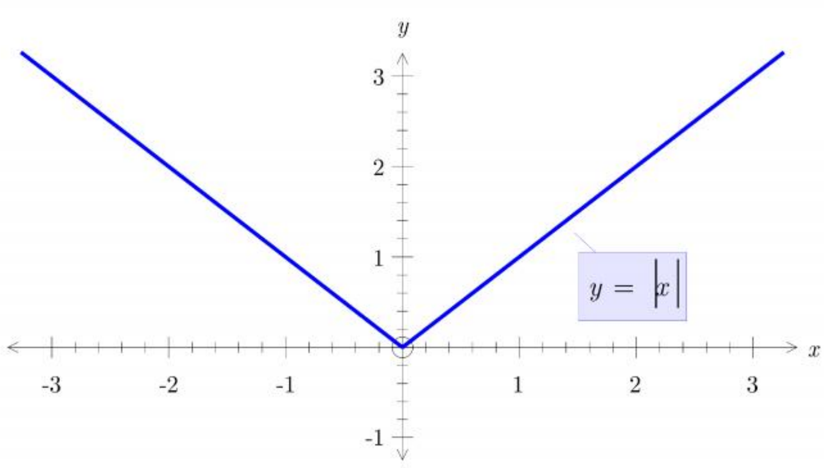


Figure 4

What do you notice about the graph of $y = |x|$ *?*

Symmetrical about the y axis, that is $|-x| = |x|$. We will prove this result later in this section.

Also the graph of $y = |x|$ is above the x axis and only touches it at $x = 0$. Hence the modulus of a real number is always positive or zero and we can write this as an inequality

(4.12) $$|x| \geq 0$$

What are the values of x *which satisfies* $|x| = 2$ *?*

By the above definition

(4.11) $$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

we have $|x| = 2$ if $x = 2$ or $x = -2$. Therefore $|x| = 2$ has the solutions $x = 2$ or $x = -2$.

The modulus function is also called the **distance function** or the **absolute function**.

The geometrical interpretation of the modulus function is the distance on the real number line. For example $|x|$ is the distance from x to zero. Consider the above example.

We can visualize $|x| = 2$ as the distance of x from zero is 2 and this is represented by:

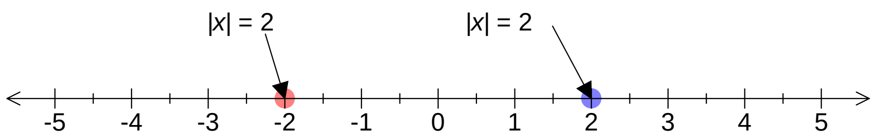


Figure 5

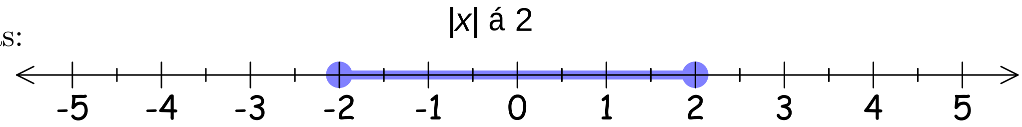
The equation $|x| = 2$ has the solution $x = 2$ or $x = -2$.

E2 Inequalities of the Modulus Function

Visualizing the modulus function as the distance function is particularly useful for inequalities.

What does $|x| \le 2$ *represent?*

The distance of x from zero is less than or equal to 2 and can be illustrated as:



What is the difference between $x \le 2$ *and* $|x| \le 2$ *?*

Figur

epresents all the real numbers less than or equal to 2. The notation $|x| \le 2$ represents all the real numbers between $x = -2$ to $x = 2$. This is the set of all real numbers between -2 to 2 and is denoted by $\{x \in \mathbb{R} \mid -2 \le x \le 2\}$ which is the interval $[-2,\ 2]$.

What values of x *satisfy* $|x| < 2$ *?*

This can be illustrated by:

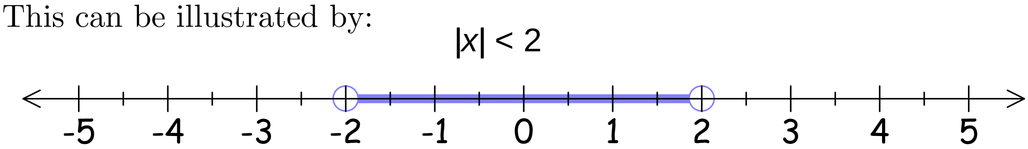


Figure 7

The values of x which satisfy $|x| < 2$ is the set $\{x \in \mathbb{R} \mid -2 < x < 2\}$ which is the open interval $]-2,\ 2[$. This is all the real numbers between -2 to 2 but **excluding** $x = 2$ and $x = -2$.

Example 1

Determine the set of real numbers, x, such that $|x| < 0.5$.

Solution.

What does $|x| < 0.5$ *mean in terms of the distance function?*

Means that the distance of x from zero is less than 0.5 and is illustated by:

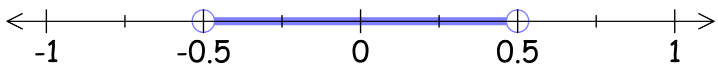


Figure 8

$|x| < 0.5$

The set of real numbers which satisfy $|x| < 0.5$ is $\{x \in \mathbb{R} \mid -0.5 < x < 0.5\}$.

We can solve general inequalities such as $|x| < a$ where $a > 0$ by algebraic means. *What does* $|x| < a$ *mean?*

The distance x from zero is less than a and can be illustrated on the real line as:

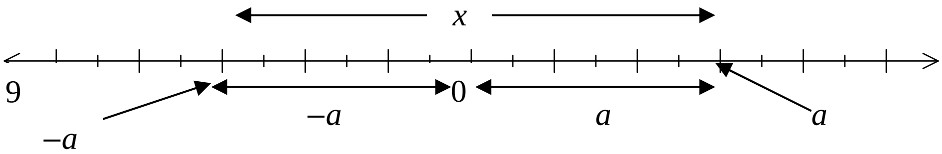


Figure 9

$|x| < a$ means that x lies between $-a$ to a and is denoted by $-a < x < a$ which we can write in symbolic form as

(4.13) $$|x| < a \quad \Leftrightarrow \quad -a < x < a$$

We will prove a version of this result later in this section.

Example 2

Determine the set of real numbers, x, such that $|x-3| < 5$.

Solution.

How can we interpret $|x-3| < 5$ *as distance on the real number line?*

This inequality, $|x-3| < 5$, means that the distance from 3 is less than 5 and we can illustrate this by:

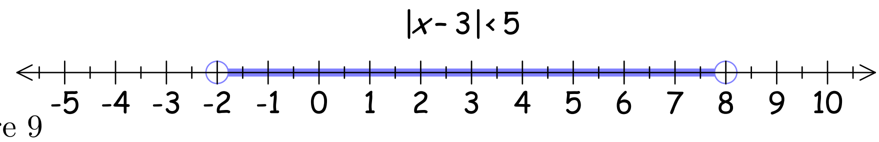


Figure 9

Hence the set of real numbers which satisfy the given inequality $|x-3| < 5$ is $\{x \in \mathbb{R} \mid -2 < x < 8\}$ which is the open interval $]-2,\ 8[$.

We can also solve this algebraically as follows:

$$\begin{aligned} &|x-3| < 5 && \\ -5 < {} & x-3 < 5 && [\text{By (4.13)}] \\ -5+3 < {} & x < 5+3 && [\text{Adding 3}] \\ -2 < {} & x < 8 && [\text{Simplifying}] \end{aligned}$$

Hence all the real numbers between -2 to 8 satisfies the given inequality $|x-3| < 5$.

Example 3

Solve the inequality $|x+3| < 5$.

Solution.

How can we interpret $|x+3|<5$ *as distance on the real number line?*

We can rewrite $|x+3|<5$ as $|x-(-3)|<5$. That is $|x+3|=|x-(-3)|<5$.

This $|x+3|<5$ means that the distance from -3 is less than 5 and we can illustrate this by:

$|x+3|<5$

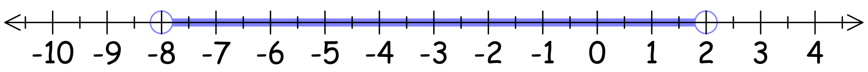


Figure 10

We can show this algebraically:

$$
\begin{aligned}
|x+3| &< 5 \\
-5 < x+3 &< 5 && [\text{By (4.13)}] \\
-5-3 < x &< 5-3 && [\text{Subtracting } 3] \\
-8 < x &< 2 && [\text{Simplifying}]
\end{aligned}
$$

Example 4

Solve the inequality $|2x+3|<5$.

Solution.

We have

$$
\begin{aligned}
|2x+3| &< 5 \\
-5 < 2x+3 &< 5 && [\text{By (4.13)}] \\
-5-3 < 2x &< 5-3 && [\text{Subtracting } 3] \\
-8 < 2x &< 2 && [\text{Simplifying}] \\
-4 < x &< 1 && [\text{Dividing by } 2]
\end{aligned}
$$

E3 Properties of the Modulus Function

This subsection is more demanding because the proofs of the results require you to thoroughly understand the definition of the modulus function and inequalities of real numbers as discussed in section D.

Proposition (4.14). For all $x \in \mathbb{R}$ we have

(a) $|x^2| = x^2$

(b) $|x|^2 = x^2$

Proof.

(a) Since $x^2 \geq 0$ therefore by definition (4.11) we have our required result, $\left|x^2\right| = x^2$.

■

(b) If $x \geq 0$ then $|x|^2 = |x||x| = xx = x^2$.

If $x < 0$ then

$$\begin{aligned} |x|^2 &= |x||x| \\ &= (-x)(-x) \qquad \left[\text{Because } x<0 \text{ therefore } |x| = -x\right] \\ &= x^2 \end{aligned}$$

■

Proposition (4.15). Let $a \in \mathbb{R}$ and $a \geq 0$, then $|x| \leq a \quad \Leftrightarrow \quad -a \leq x \leq a$.

Comment. *How do we prove this proposition?*

Since we have the implication going both ways, $\Leftrightarrow$, in our proposition so we have to prove $\Rightarrow$ and its converse $\Leftarrow$. Hence the proof below is split into two halves, first we prove $|x| \leq a \quad \Rightarrow \quad -a \leq x \leq a$ and then we prove $-a \leq x \leq a \quad \Rightarrow \quad |x| \leq a$.

Remember our work on mathematical logic from chapter 1, for a $P \Rightarrow Q$ proof we assume P and deduce Q. Similarly for $Q \Rightarrow P$ proof we assume Q and deduce P.

Proof. $(\Rightarrow)$. Assume $|x| \leq a$. Required to prove $-a \leq x \leq a$.

If $x \geq 0$ then $|x| = x$ so we have $|x| = x \leq a$. *Why?*

Because by assumption we have $|x| \leq a$.

If $x < 0$ then $|x| = -x$ so we have

$$\begin{aligned} |x| = -x &\leq a \\ x &\geq -a \qquad \left[\text{Multiplying by } -1\right] \end{aligned}$$

Remember multiplying by a negative number changes the inequality.

We have $x \leq a$ and $x \geq -a$ which means that x lies between $-a$ to a and is normally written as $-a \leq x \leq a$. Hence we have proven $\Rightarrow$.

$(\Leftarrow)$. Now lets go the other way. Assume $-a \leq x \leq a$. Required to prove $|x| \leq a$. We can use proof by contradiction. Suppose $|x| > a$. We first consider $x \geq 0$ (positive or zero) and then $x < 0$ (negative).

If $x \geq 0$ then $|x| = x > a$. Contradiction! *Why?*

Because by assumption we have $x \le a$ (x is less than or equal to a) but now $x > a$ (x is greater than a). Our supposition $|x| > a$ must be false so therefore $|x| \le a$.

Similarly if $x < 0$ then $|x| = -x > a$. Multiplying this, $-x > a$, by -1 changes the inequality and we have

$$x < -a \text{ which is equivalent to } -a > x$$

Contradiction! *Why?*

Because by assumption we have $-a \le x$ ($-a$ is less than equal to x) and now by deduction $-a > x$ ($-a$ is greater than x). Our supposition $|x| > a$ must be false, therefore $|x| \le a$.

■

Note that this proposition $|x| \le a \quad \Leftrightarrow \quad -a \le x \le a$ is similar to the stated earlier result:

(4.13) $$|x| < a \quad \Leftrightarrow \quad -a < x < a$$

but includes equality.

Proposition (4.16). For all $x \in \mathbb{R}$ we have $-|x| \le x \le |x|$.

Proof. Using the above proposition (4.15) with $a = |x|$ that is $|x| \le |x|$ implies

$$-|x| \le x \le |x|$$

■

Proposition (4.17). For all $x \in \mathbb{R}$ and $y \in \mathbb{R}$ we have

$$|xy| = |x||y|$$

Proof. We consider the 4 possible cases:

(1) If both $x \ge 0$ and $y \ge 0$ then $xy \ge 0$ and we have

$$|x||y| = xy = |xy|$$

(2) If both $x < 0$ and $y < 0$ then $xy > 0$ and we have

$$\begin{aligned} |x||y| &= (-x)(-y) \\ &= xy = |xy| \end{aligned}$$

Remember by the definition of the modulus function if $x < 0$ and $y < 0$ then $|x| = -x$ and $|y| = -y$ respectively.

(3) If $x > 0$ and $y < 0$ then $xy < 0$ and we have

$$
\begin{aligned}
|x||y| &= x(-y) && \left[\text{Because } y<0 \text{ so } |y|=-y\right] \\
&= -xy \\
&= |xy| && \left[\text{Because } xy<0 \text{ so } |xy|=-xy\right]
\end{aligned}
$$

(4) If $x<0$ and $y>0$ then $xy<0$ and we have

$$
\begin{aligned}
|x||y| &= (-x)\,y && \left[\text{Because } x<0 \text{ so } |x|=-x\right] \\
&= -xy \\
&= |xy| && \left[\text{Because } xy<0 \text{ so } |xy|=-xy\right]
\end{aligned}
$$

■

Proposition (4.18). For all $x \in \mathbb{R}$ we have

$$|-x| = |x|$$

Proof. We use the above proposition (4.17) to prove this result:

$$
\begin{aligned}
|-x| &= |(-1)\,x| \\
&= |-1||x| && \left[\text{By (4.17) } |xy|=|x||y|\right] \\
&= 1|x| && \left[\text{Because } |-1|=1\right] \\
&= |x|
\end{aligned}
$$

■

Triangle Inequality (4.19).

Let both x and y be real numbers. Then we have the **important** inequality

$$|x+y| \le |x| + |y|$$

Proof.

We consider $|x+y|^2$ and prove this is less than or equal to $(|x|+|y|)^2$ and then take the square root of both sides. Hence we have

$$
\begin{aligned}
|x+y|^2 &= (x+y)^2 && \left[\text{Because } (x+y)^2 \ge 0\right] \\
&= x^2+y^2+2xy && [\text{Expanding}] \\
&\le x^2+y^2+2|xy| && \left[\text{Because by (4.16) } xy \le |xy|\right] \\
&= |x|^2+|y|^2+2|x||y| && \left[\text{Because by (4.17) } |xy|=|x||y|\right] \\
&= (|x|+|y|)^2 && [\text{Factorizing}]
\end{aligned}
$$

We have $|x+y|^2 \le (|x|+|y|)^2$. Taking the square root keeps the same inequality, therefore we have our required result, $|x+y| \le |x|+|y|$.

■

You are asked to provide another proof of this triangular inequality in the Exercises.

Next we look at $|x-a| < \varepsilon$ where $\varepsilon > 0$.

Let a be a real number and $\varepsilon > 0$. In general the inequality $|x-a| < \varepsilon$ means the distance from a is less than £ and can be illustrated on the real number line as:

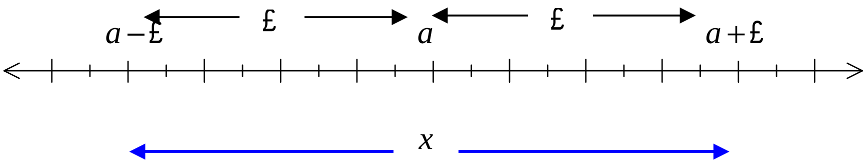


Figure 11

Like the last section the Greek letter £ is used to represent a small positive real number. Do not let this Greek letter put you off. The symbol £ is generally used in mathematics to represent small positive real numbers.

One of the greatest mathematician of 20th Century Paul Erdos used to call young children epsilons.

Paul Erdos was born to a Hungarian Jewish family in Budapest in 1913 and died in Warsaw at a conference in 1996.

Figure 12

We can solve the given inequality, $|x-a| < £$, by algebraic means:

$$|x-a| < £$$

$$-£ < x-a < £ \qquad [\text{By } (4.13)]$$

$$a-£ < x < a+£ \qquad [\text{Adding } a]$$

The set of real numbers, x, which lie between $a-£$ to $a+£$ is called the £ - **neighbourhood** of a.

Proposition (4.20). Let a be a real number and $£ > 0$. If for every $£ > 0$ we have

$$|x-a| < £$$

then $x = a$.

Proof. By (4.12) we know the modulus function is greater than or equal to zero, $|x-a| \geq 0$. Because for every $£ > 0$ we have

$$0 \leq |x-a| < £$$

therefore by proposition (4.9) we have

$$0 \le |x-a| < £ \text{ implies } |x-a| = 0$$

$|x-a| = 0$ implies $x-a=0$. Hence $x=a$ which is our result.

■

What does proposition (4.20) mean?

Proposition (4.20) means that if x can be made as close as we please to a then $x=a$.