SECTION C **Inverse Functions**

By the end of this section you will be able to

- show that a given function is bijective
- find the inverse function

You need to know your work on injections and surjections from section B thoroughly to understand this section. It is a much more difficult section than the previous two (3A and 3B).

C1 Bijective Functions

Guess what the term bijective function means?

A function which is both injective (one to one) and surjective (onto). A **bijective** function is also called a **bijection**. Hence a bijection is a function from a set A (domain) to a set B (codomain) which is both **injective** and **surjective**. Therefore a bijective function is both **one to one** and **onto**.

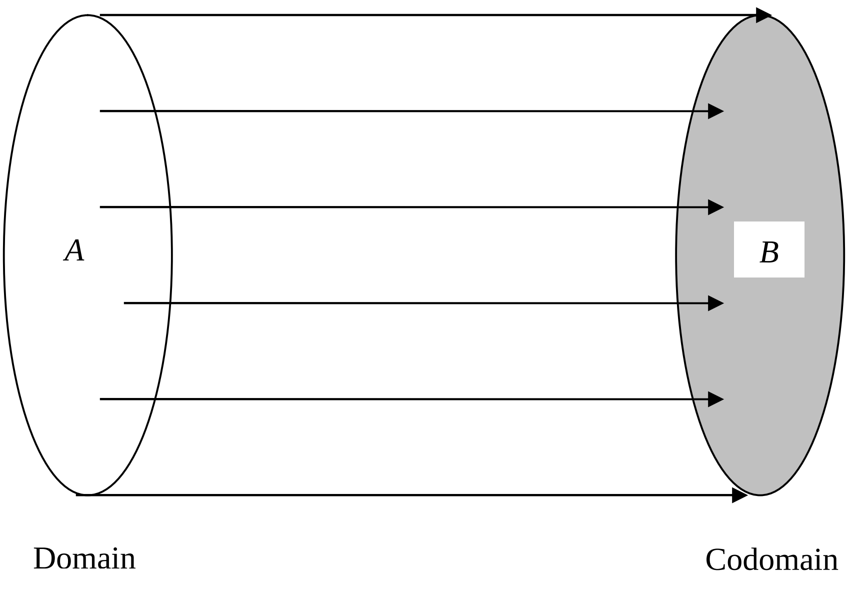

Fig 16

Fig 16 shows a bijective function.

Example 17

Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=2x+1$. Show that f is a bijective function.

Solution

What do we need to prove?

We have to prove that the given function $f(x)=2x+1$ is **one to one** and **onto**. First we show that the function is one to one that is f is injective. We show

$$f(x)=f(y) \quad \text{gives} \quad x=y$$

How?

Well $f(x)=2x+1$ and $f(y)=2y+1$ equating these we have

$$2x+1=2y+1$$
$$2x=2y \quad \text{which gives} \quad x=y$$

Hence the given function f is one to one. Next we show f is onto (surjective). *How?*

Let $y=f(x)$ where y is in the codomain and then find an x in the domain.

$$y=2x+1$$
$$x=\frac{y-1}{2} \qquad \text{[Transposing]}$$

Since $y\in\mathbb{R}$ (because it is in the codomain) which means that y is a real number, therefore $x=\frac{y-1}{2}$ is also a real number so we have found an x in the domain, $\mathbb{R}$. Hence f is onto (surjective).

Since the given function $f(x)=2x+1$ is both, one to one and onto, we conclude that it is a bijective function.

One of the difficulties in this section is it is easy to get confused with all the notation used to show a given function is bijective. For a function to be injective (one to one) we consider elements x and y in the domain:

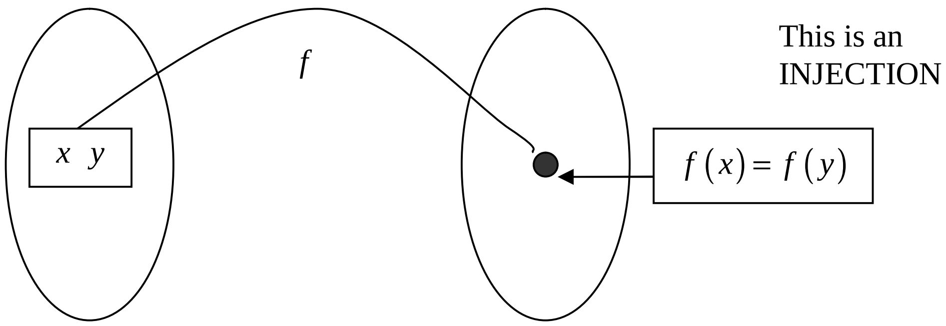

Fig 17

To show surjection (onto) we consider y to be in the codomain and then find an x in the domain which corresponds to this y via the function f.

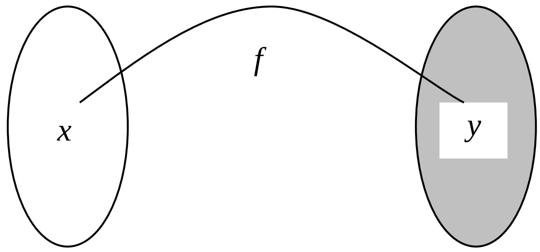

Fig 18

From the last section you might have noticed that any function can be made surjective (onto) by changing the codomain of that function. Similarly we can change the domain to make the function injective (one to one).

In the next example we use the method of 'completing the square' to find an appropriate codomain for the function to be surjective. If you have forgotten this method of 'completing the square' or you have not covered it, then see Appendix B.

Example 18

Let $f:\mathbb{R}^+ \to \mathbb{R}$ be defined by $f(x)=x^2-4x+5$. What is the largest codomain so that f is surjective (onto)?

Solution

We apply the technique of 'completing the square' on $f(x)=x^2-4x+5$.

$$x^2-4x+5=(x-2)^2-4+5 \qquad \left[\text{Because } (x-2)^2=x^2-4x+4\right]$$

$$=(x-2)^2+1 \qquad [\text{Because } -4+5=1]$$

Hence we have

$$f(x)=x^2-4x+5$$
$$=(x-2)^2+1$$

What do we know about the square part, $(x-2)^2$ *?*

It is always positive or zero because squaring any real number does **not** give a negative number. Hence

$$(x-2)^2 \geq 0$$

Therefore $f(x)=(x-2)^2+1\geq 0+1=1$. Hence $f(x)\geq 1$. Therefore the largest codomain is going to be the set of real numbers which are greater than or equal to 1. *How do we write this?*

$$B=\{x\in\mathbb{R} \mid x\geq 1\}.$$

Next we prove that the function given by the formula in Example 18 is bijective for the appropriate domain and codomain.

Example 19

Let $A=\{x\in\mathbb{R} \mid x\geq 2\}$ and $B=\{x\in\mathbb{R} \mid x\geq 1\}$. Let $f:A\to B$ be defined by $f(x)=x^2-4x+5$. Prove that f is a bijective function.

Solution

We have to prove that the given function $f(x)=x^2-4x+5$ is **one to one** and **onto**. First we show that the function is one to one that is f is injective. We need to show

$$f(x)=f(y) \quad \text{gives} \quad x=y$$

How?

$$\begin{aligned}
x^2-4x+5 &= y^2-4y+5 && [f(x)=f(y)] \\
x^2-y^2-4x+4y &= 0 && [\text{Rearranging}] \\
(x-y)(x+y)-4(x-y) &= 0 && [\text{Remember } x^2-y^2=(x-y)(x+y)] \\
(x-y)[(x+y)-4] &= 0 && [\text{Factorizing}] \\
(x-y)=0 \text{ or } x+y-4 &= 0 \\
x=y \text{ or } x &= 4-y && [\text{Solving for } x]
\end{aligned}$$

Consider the solution $x=4-y$. Remember y is a member of the domain, $A=\{x\in\mathbb{R} \mid x\geq 2\}$, which means it is a real number greater than or equal to 2. If $y=2$ then $x=4-2=2$ and we have $x=y=2$. If $y>2$ then $x=4-y<2$ which means x cannot be in the domain, $A=\{x\in\mathbb{R} \mid x\geq 2\}$.

Why not?

Because $A=\{x\in\mathbb{R} \mid x\geq 2\}$ and so only contains real numbers ≥ 2. Hence we dismiss the solution $x=4-y$.

Therefore we only have the solution $x=y$. *What can we conclude from this?*

The given function is one to one.

What else do we need to show for f to be bijective?

The given function $f(x)=x^2-4x+5$ is onto. *How?*

The procedure outlined in the last section – that is let $y=f(x)$ be in the codomain and then find an x in the domain which corresponds to this y.

How?

By solving $y=f(x)$ for x. We have

$$y = f(x)$$
$$= x^2 - 4x + 5 = (x-2)^2 + 1 \quad \text{[From Example 18]}$$

How do we find x *from* $y = (x-2)^2 + 1$*?*

By transposing

$$y = (x-2)^2 + 1$$
$$y - 1 = (x-2)^2 \quad \text{[Subtracting 1]}$$
$$\pm\sqrt{y-1} = x - 2 \quad \text{[Taking Square Root]}$$
$$\pm\sqrt{y-1} + 2 = x \quad \text{[Adding 2]}$$

Remember $x = 2 \pm \sqrt{y-1}$ means $x = 2 + \sqrt{y-1}$ or $x = 2 - \sqrt{y-1}$. Choose $x = 2 + \sqrt{y-1}$. *Why?*

Because the domain is $A = \{x \mid x \in \mathbb{R} \text{ and } x \geq 2\}$, real numbers ≥ 2, and we need to locate an x in the domain for f to be onto. *Is* x *in the domain?*

Yes because y is in the codomain and therefore $y \geq 1$ so $x = 2 + \sqrt{y-1}$ is a real number greater than or equal to 2. We have

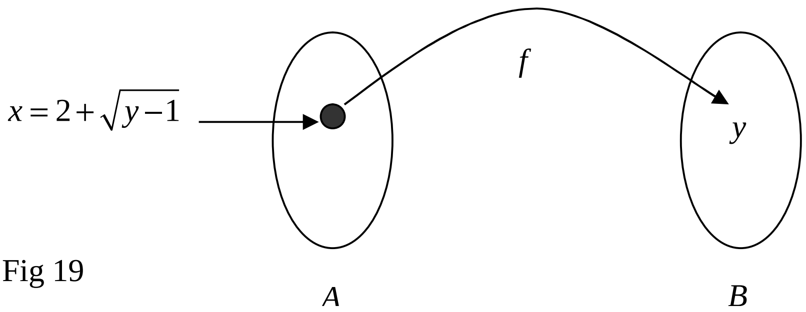

Fig 19

Hence $x \in A = \{x \in \mathbb{R} \mid x \geq 2\}$ which means we have found an x in the domain. We conclude that f is onto.

Since f is one to one (injective) and onto (surjective) therefore f **is bijective**.

C2 Inverse Functions

What does the term inverse function mean?

Inverse in everyday life means opposite or reverse. What is the inverse of

A. Tying shoe laces?
B. Multiplying by 5?
C. Adding 3 and then multiplying by 43?

The inverses are

A. Untying shoe laces.

B. Dividing by 5.

C. Dividing by 43 and then subtracting 3.

Inverse functions simply unlock what you have done. Note the inverse of part C. You undo the last operation first.

For example let $f: A \to B$ be a function such that $f(x) = y$. *What does this mean?*

It means that the function f takes x to y. W*hat do you think the inverse function does?*

The inverse function must take y back to x.

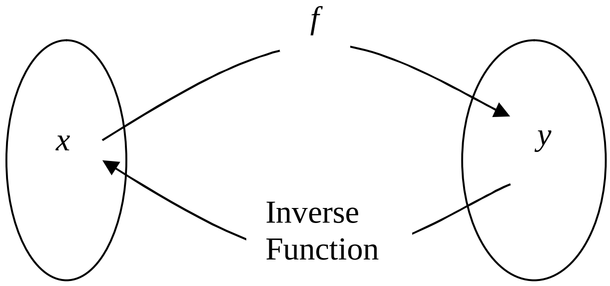

Fig 20

For example if $f(2) = -100$ then the inverse function say g must assign -100 to 2, that is $g(-100) = 2$.

Consider another example if $f(\text{France}) = \text{Paris}$ then the inverse function say g must take Paris back to France, that is $g(\text{Paris}) = \text{France}$.

If $f(\text{Blue}) = \text{Green}$ then the inverse function say g must take Green back to blue, $g(\text{Green}) = \text{Blue}$. The definition of the inverse function is:

Definition (3.4)

Let $f: A \to B$ be a function. If for an arbitrary x in A we have $f(x) = y$ in B then the function $g: B \to A$ given by

$$g(y) = x$$

is called the **inverse function** of f. The inverse function is normally denoted by f^{-1}.

We have

f

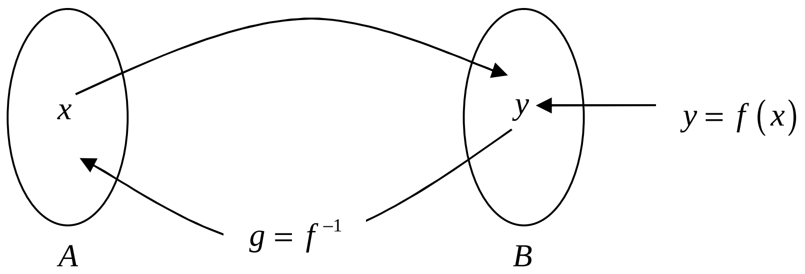

Fig 21

For brevity we say the function f^{-1} is the **inverse** of f. *What is the domain and codomain of* f^{-1} *?*

Since $f : A \to B$ therefore $f^{-1} : B \to A$ because the inverse function reverses f so the domain is B and codomain is A of f^{-1}.

An important proposition in functions is that the function $f : A \to B$ has an inverse if and only if the function f is bijective. We prove this proposition later in this section. This result means that bijective functions and inverse functions are equivalent. To see why consider the following examples.

Let $f : \mathbb{R} \to \mathbb{R}^{+} \cup \{0\}$ be defined by $f(x) = x^2$. Then

$$f(2) = f(-2) = 4$$

Hence f is **not** injective (one to one) but is surjective (onto). *But where does the inverse function,* f^{-1}*, take 4? What is the value of the inverse function at* 4 *?*

Is it 2 or -2 *?*

There is **no** unique value of $f^{-1}(4)$. We do **not** necessarily go back to our initial value.

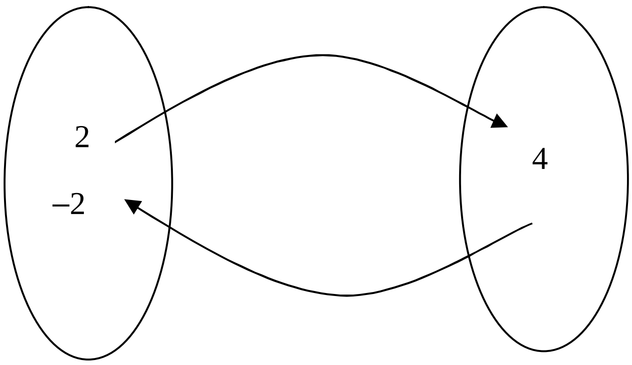

Fig 22

We need the function to be injective (one to one) for it to have an inverse. Otherwise which value would 4 go back to?

Now consider a function which is injective but **not** surjective.

Let $f : \mathbb{R}^{+} \to \mathbb{R}$ be defined by

$$f(x) = x^2$$

Then f is injective but **not** surjective (onto). *What is* $f^{-1}(-3)$ *equal to?*
There is **no** x in the domain, $\mathbb{R}^+$, such that

$$f(x) = x^2 = -3$$

This means that $f^{-1}(-3)$ is **not** defined as we are ONLY allowed to have real numbers.

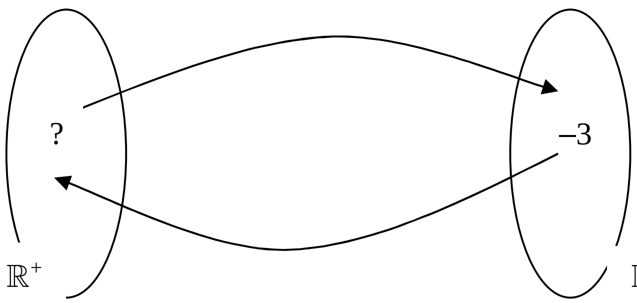

Fig 23

In fact non of the negative numbers are assigned by the function f. Hence we need f to be surjective (onto) for f to have an inverse function. Otherwise -3 cannot go back to any value in the domain.
Next we prove this important result that the function f has an inverse if and only if it is bijective. This is a difficult proof to follow. It is easy to get confused with all the similar statements. It seems that we are presenting a circular argument to prove our result but we are not. Read the proof carefully.

Proposition (3.5)
Let $f : A \to B$ be a function. Then f has an inverse $\Leftrightarrow$ f is bijective.

Proof. Remember a $\Leftrightarrow$ proof means we have to prove the result both ways, that is if f has an inverse then f is bijective and also if f is bijective then f has an inverse.
$(\Rightarrow)$. Assume f has an inverse denoted by g say. *What do we need to prove for this part?*
Need to prove f is injective and surjective. *How do we show* f *is injective?*
Required to prove

$$f(x) = f(y) \quad \text{gives} \quad x = y$$

Let $f(x) = f(y) = z$. f

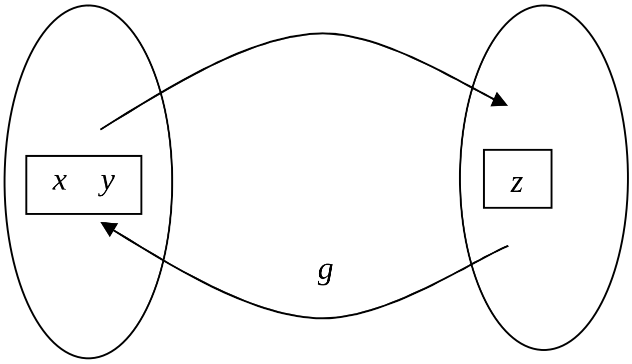

Fig 24

Then by the definition of the inverse function

$$g(z)=x \text{ and } g(z)=y$$

Hence $x=g(z)=y$. Therefore we have shown that the given function f is injective.

What else do we need to show?

The function f is surjective. *How?*

By the procedure outlined in the last section, that is assume $y=f(x)$ is a member of the codomain, B, and then show that there is an x in the domain, A, which corresponds to this y.

Let $y=f(x)$ then by the definition of the inverse function $g(y)=x$. Hence we have located an x in the domain such that $y=f(x)$ therefore the given function f is surjective.

Since f is injective and surjective therefore f is bijective.

$(\Leftarrow)$. Now let's go the other way, that is assume f is bijective and deduce that f has an inverse function.

Let the function f be bijective. Required to prove that there is a function $g: B \to A$ which reverses what the function f does.

Define $g: B \to A$ by

$$g(y)=x$$

where x is the element in A such that $f(x)=y$.

We need to prove that such a g is a function. For $g: B \to A$ to be a function we need every element in B assigned to only one element in A because this is the definition of a function. (Everything in B has only one destination).

Let $y \in B$ then there is an x in A such that $f(x)=y$. *Why?*

Because f is onto (surjective) so every element in B is assigned to.

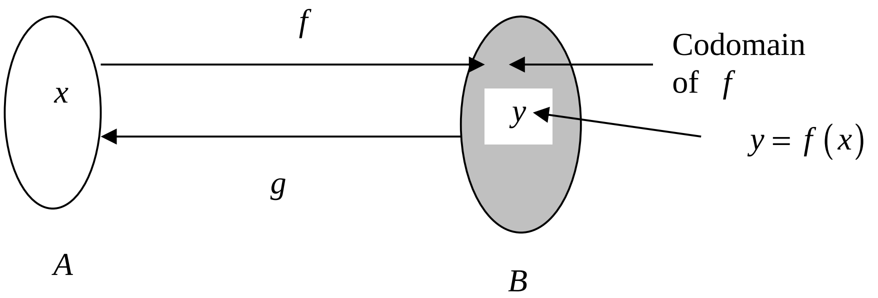

Fig 25

Therefore everything in B gets assigned by g to A.

Need to prove that g assigns elements in B to only one element in A.

We have $f(x)=y$. Suppose $g(y)=z$ then by our definition of g we have $f(z)=y$ and therefore

$$f(x)=y=f(z)$$

Since f is injective then by

(3.2) A function f is **injective** $\Leftrightarrow$ $f(x)=f(y) \Rightarrow x=y$

we have $x=z$.

Hence g assigns everything in B to a unique element in A. Therefore g is a function and is the inverse function of f.

■

Try reading through the proof again and see if you can do parts of the proof without looking. Hence a function f has an inverse if and only if it is bijective.

In Example 17 we showed the following function is bijective:

$f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=2x+1$. *How do we find the inverse of this function?*

The procedure to find the inverse function is the same technique as to show a given function is surjective. Let $y=f(x)$ then solve this equation for *x.*

Example 20

Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=2x+1$. Determine the inverse function $f^{-1}(x)$.

<u>Solution</u>

Let $y=2x+1$ and solve this equation for x.

$$y=2x+1$$

$$x=\frac{y-1}{2}$$

What is $f^{-1}(x)$ *equal to?*

$f^{-1}(x)$ is a function of x therefore we have

$$f^{-1}(x)=\frac{x-1}{2} \qquad \left[\begin{array}{l}\text{Replacing the } y \text{ by } x\\ \text{in the above}\end{array}\right]$$

Remember $f^{-1}(x)\neq\dfrac{y-1}{2}$ [Not Equal] because the inverse function is a function of x.

We have actually found the inverse function $f^{-1}(x)$ in Example 17 when we showed the function was surjective.

Example 21

Let $A=\{x\in\mathbb{R} \mid x\geq 2\}$ and $B=\{x\in\mathbb{R} \mid x\geq 1\}$. Let $f:A\to B$ be defined by $f(x)=x^2-4x+5$. Find the inverse function $f^{-1}(x)$.

Solution

We have already shown that the given function, $f(x)=x^2-4x+5$, is bijective in Example 19. Remember the inverse function is given by letting $y=f(x)$ and then solving this equation for x which is what we used to show the given function is surjective (onto). From Example 19 we have

$$x=\sqrt{y-1}+2$$

by solving $y=f(x)$ for x.

What is the inverse function, $f^{-1}(x)$, *equal to?*

$f^{-1}(x)$ is a function of x therefore we have

$$f^{-1}(x)=\sqrt{x-1}+2 \qquad \left[\begin{array}{l}\text{Replacing } y \text{ by } x\\ \text{in the above}\end{array}\right]$$

Example 22

Let $A=\{x\in\mathbb{R} \mid x\neq 1\}$ and $B=\{x\in\mathbb{R} \mid x\neq 3\}$ and $f:A\to B$ be defined by

$$f(x)=\frac{3x}{x-1}$$

Find the inverse function $f^{-1}(x)$.

Solution

We can show the given function is bijective and in the process find the inverse function. *How do we show the given function is bijective?*

First show it is injective (one to one) by letting $f(x) = f(y)$ and then deduce $x = y$:

We have $f(x) = \dfrac{3x}{x-1}$ and $f(y) = \dfrac{3y}{y-1}$. Equating these and simplifying

$$\begin{aligned}
\frac{3x}{x-1} &= \frac{3y}{y-1} & \\
3x(y-1) &= 3y(x-1) & &\left[\text{Multiplying through by } (y-1)(x-1)\right] \\
3xy - 3x &= 3yx - 3y & &\left[\text{Expanding Brackets}\right] \\
-3x &= -3y & &\left[\text{Simplifying}\right] \\
x &= y & &\left[\text{Dividing through by } -3\right]
\end{aligned}$$

Hence f is injective. *What else do we need to show?*

Required to prove that f is surjective. *How?*

By considering y to be in the codomain and finding an x in the domain such that $y = f(x)$. We need to solve $y = f(x)$ for x.

$$\begin{aligned}
y &= \frac{3x}{x-1} & \\
y(x-1) &= 3x & &\left[\text{Multiplying by } (x-1)\right] \\
yx - y &= 3x & &\left[\text{Opening Brackets}\right] \\
yx - 3x &= y & &\left[\text{Collecting like terms}\right] \\
x(y-3) &= y & &\left[\text{Factorizing}\right] \\
x &= \frac{y}{y-3} &
\end{aligned}$$

Now $x = \dfrac{y}{y-3}$ is in the domain provided $y \neq 3$. Since y is in the codomain, $B = \left\{x \in \mathbb{R} \,\middle|\, x \neq 3\right\}$, therefore $y \neq 3$. Hence we have found an x in the domain such that $y = f(x)$. Therefore f is surjective.

Hence the given function is bijective.

What is the inverse function $f^{-1}(x)$ equal to?

$$f^{-1}(x) = \frac{x}{x-3} \qquad \left[\begin{array}{l}\text{Replacing the } y \text{ by } x \\ \text{in the above}\end{array}\right]$$

Remember this is same as showing that the given function is surjective.

SUMMARY

A function f is a **bijective** function if it is both injective (one to one) and surjective (onto).

Let $f: A \to B$ be a function. If $f(x) = y$ then the inverse function $f^{-1}: B \to A$ is given by $f^{-1}(y) = x$.

A function f has an inverse if and only if f is bijective.

To find the inverse function we only have to show the function is bijective and in the process we find the inverse function by letting $y = f(x)$ and then solve this equation for x.