Section C: Properties of Uniform Convergence and Series of Functions
By the end of this section you will be able to • understand the relationship between uniform convergence and continuity • understand what is meant by a series of functions • test a series for uniform convergence

For **pointwise** convergence we fix a value for *x* and then choose a N_0. This means that the natural number N_0 depends on **both** the given $£ > 0$ and *x*.

For **uniform** convergence $f_n(x)$ must be uniformly close to the limiting function $f(x)$ for **all** x in the domain and $\forall n \ge N_0$. Thus N_0 depends on $£ > 0$ only (**not** x).

C1 Continuity of Functions

If a sequence of functions $(f_n(x))$ *converges uniformly to limiting function* $f(x)$ *and each function* $f_n(x)$ *is continuous at a point in the domain then is the limiting function also continuous at the same point?*

Yes. Next we prove this result.

Proposition (3.5). Let the sequence of functions $(f_n(x))$ converge **uniformly** to a function $f(x)$ in the domain *D*. If each $f_n(x)$ is continuous at a point *c* in the domain *D* then the limit function $f(x)$ is also continuous at point *c*.

How do we prove this result?

We need to know the definitions of continuity at point and uniform convergence. From our work on continuous functions we have the definition:

A function *f* is **continuous** at a point *c* $\Leftrightarrow$ for every $£ > 0$ there exists a $¢ > 0$ such that

$$|x - c| < ¢ \quad \Rightarrow \quad |f(x) - f(c)| < £$$

We use this definition to prove the given result.

Proof.

Let $£ > 0$ be given. Since $f_n(x)$ converges uniformly to $f(x)$ then by definition (3.3) on page 17:

The sequence of functions $\left(f_n(x)\right)$ converges **uniformly** to a function $f(x)$ in the domain *D* $\Leftrightarrow$ for every x in *D* and for every $£ > 0$ there exists a natural number N_0 (depending only on £) such that $\left|f_n(x)-f(x)\right| < £$ provided $n \geq N_0$.

Take $n = N_0$ then

$$\left|f_{N_0}(x)-f(x)\right| < \frac{£}{3} \qquad (*)$$

This is true for all x in *D*. This means the result (*) also holds for the point *c* in *D*:

$$\left|f_{N_0}(c)-f_{N_0}(c)\right| < \frac{£}{3} \qquad (**)$$

We are given that for each *n* we have $f_n(x)$ is continuous at a point *c*. In particular $f_{N_0}(x)$ is continuous at a point c. This means there is a $¢ > 0$ such that

$$|x-c| < ¢ \quad \Rightarrow \quad \left|f_{N_0}(x)-f_{N_0}(c)\right| < \frac{£}{3} \qquad (***)$$

Consider the distance $\left|f(x)-f(c)\right|$ where $|x-c| < ¢$ we have:

$$\begin{aligned} \left|f(x)-f(c)\right| &= \left|f(x)-f_{N_0}(x)+f_{N_0}(x)-f_{N_0}(c)+f_{N_0}(c)-f(c)\right| \\ &\leq \underbrace{\left|f(x)-f_{N_0}(x)\right|}_{<£/3 \text{ by } (*)} + \underbrace{\left|f_{N_0}(x)-f_{N_0}(c)\right|}_{<£/3 \text{ by } (***)} + \underbrace{\left|f_{N_0}(c)-f(c)\right|}_{<£/3 \text{ by } (**)} \\ &< \frac{£}{3}+\frac{£}{3}+\frac{£}{3} = £ \end{aligned}$$

We have $|x-c| < ¢$ implies that $\left|f(x)-f(c)\right| < £$. By the above definition of continuity:

A function *f* is **continuous** at a point *c* $\Leftrightarrow$ for every $£ > 0$ there exists a $¢ > 0$ such that

$$|x-c| < ¢ \quad \Rightarrow \quad \left|f(x)-f(c)\right| < £$$

We conclude that limit function $f(x)$ is continuous at point *c*.

□

C2 Series of Functions

A series of functions denoted by $\sum f_n(x)$ is the sequence of functions which are the nth partial sum of the sequence defined by:

$$S_n(x)=\sum_{m=0}^{n} f_m(x)$$

Examples are power series and Fourier series.

An example of a series of functions is the geometric series $\sum_{n=0}^{\infty} x^n$. We can write this as:

$$\sum_{n=0}^{\infty} x^n = 1+x+x^2+x^3+x^4+\cdots$$

The partial sums are given by:

$$S_0(x)=\sum_{m=0}^{0} f_m(x)=1$$

$$S_1(x)=\sum_{m=0}^{1} f_m(x)=f_0(x)+f_1(x)=1+x$$

$$S_2(x)=\sum_{m=0}^{2} f_m(x)=f_0(x)+f_1(x)+f_2(x)=1+x+x^2$$

$$S_3(x)=\sum_{m=0}^{3} f_m(x)=1+x+x^2+x^3$$

The formal definition of series of functions is:

Definition (3.6). Let the sequence of functions $(f_n(x))$ have domain *D*. For all *x* in *D* the sequence of partial sums $S_n(x)$ for all natural numbers ($n=1,\ 2,\ 3,\ \cdots$) is given by

$$S_n(x)=\sum_{m=0}^{n} f_m(x)$$

The definitions of pointwise and uniform convergence in the case of a series means the pointwise or uniform convergence of the nth partial sums.

If there is a function $S(x)$ such that the nth partial sums $S_n(x)$ converge uniformly (respectively pointwise) to $S(x)$ as $n\to\infty$ then the series $\sum f_n(x)$ converges to $S(x)$ uniformly (respectively pointwise) on the domain *D*. We write this as

$$S(x)=\sum_{n=0}^{\infty} f_n(x)$$

C3 Uniform Convergence of Power Series

In this subsection we look at **uniform convergence** of power series. *Can you recall what we mean by a power series?*

A power series about the real number *a* is of the form

$$c_0+c_1(x-a)+c_2(x-a)^2+c_3(x-a)^3+\cdots+c_n(x-a)^n+c_{n+1}(x-a)^{n+1}+\cdots=\sum_{n=0}^{\infty} c_n(x-a)^n$$

We have investigated the convergence of such series in the last chapter. The series may converge:

1) at single point $x=a$

2) for all $x \in \mathbb{R}$
3) for a radius of convergence say *r*, which means that the interval of converges is $]a-r,\ a+r[$:

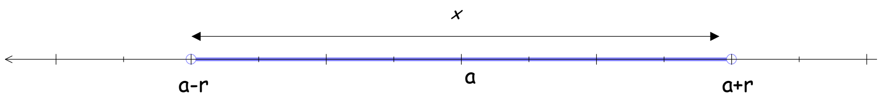

Fig 17

We have seen in the last chapter that for *x* in the interval of convergence the sum of the power series defines a function:

$$f(x)=\sum_{n=0}^{\infty}c_n(x-a)^n$$

There are two important questions that we need to address:

1) Is the function $f(x)$ continuous or differentiable?

2) Can we differentiate or integrate term by term?

This depends on whether the series is uniformly convergent. First we need to find a way for testing uniform convergence.

C4 Testing Uniform Convergence

We examine one test for uniform convergence – Weierstrass M test.

Weierstrass M-Test (3.6).

Let $(f_n(x))$ be a sequence of functions from the domain *D* to $\mathbb{R}$. Let $f: D \to \mathbb{R}$ be a function.

Let $M_1,\ M_2,\ M_3,\ \cdots$ be a sequence of positive real numbers. If

i) for all natural numbers n and for all *x* in the domain we have

$$\left|f_n(x)-f(x)\right| \le M_n$$

ii) the sequence (M_n) converges to 0

Then $(f_n(x))$ converges uniformly to $f(x)$ on the domain *D*.

How do we prove this result?

We need to use the definition of uniform convergence which was given on page 17:

Definition (3.3). The sequence of functions $(f_n(x))$ converges **uniformly** to a function $f(x)$ in the domain *D* $\Leftrightarrow$ for every x in *D* and for every $\pounds > 0$ there exists a natural number N_0 (depending only on $\pounds$) such that $|f_n(x) - f(x)| < \pounds$ provided $n \geq N_0$.

Proof.

Let $\pounds > 0$ be given. By condition (ii) we have (M_n) converges to 0. By the definition of convergence which is stated on page ??

A sequence (a_n) converges to *L* $\Leftrightarrow$ for every $\pounds > 0$ there exists a natural number N_0 (depending only on $\pounds$) such that for all $n \geq N_0$ we have $|a_n - L| < \pounds$.

Since (M_n) converges to 0, this means that there is a natural number N_0 (depending only on $\pounds$) such that for all $n \geq N_0$

$$|M_n - 0| = |M_n| \underbrace{=}_{\text{Because } M_n \text{ is positive}} M_n < \pounds$$

Applying condition (i) of the proposition for all $n \geq N_0$ and all x in the domain we have:

$$|f_n(x) - f(x)| \leq M_n < \pounds$$

By the above Definition (3.3) on uniform convergence we conclude that $(f_n(x))$ converges uniformly on the domain to $f(x)$.

□

Weierstrass M test for series is given by:

Weierstrass M-Test (3.7) for Series

Let $\sum f_n(x)$ be a power series defined on the interval of convergence. Let $M_1, M_2, M_3, \cdots$ be a sequence of positive real numbers. If

i) for all natural numbers n and for all *x* in the interval of convergence we have

$$|f_n(x)| \leq M_n$$

ii) $\sum M_n$ converges

Then $\sum f_n(x)$ converges uniformly and absolutely in the interval of convergence.

Example 14

Show that $\sum_{n=1}^{\infty}\left(\frac{\cos(nx)}{n^2}\right)$ is uniformly and absolutely convergent on $\mathbb{R}$.

Solution

How do we prove this result?

Apply Weierstrass M-Test given above. In this case $f_n(x)=\frac{\cos(nx)}{n^2}$. We need to find positive real numbers M_n which are greater than $\left|\frac{\cos(nx)}{n^2}\right|$. We have

$$\left|\frac{\cos(nx)}{n^2}\right|=\frac{|\cos(nx)|}{n^2}\le\frac{1}{n^2}\qquad\left[\text{Because }|\cos(nx)|\le 1\right]$$

We take our positive real numbers $M_n=\frac{1}{n^2}$. We know from the last chapter that

$$\sum M_n=\sum\frac{1}{n^2}\text{ converges by the } p-\text{test}$$

(The p – test is $\sum\frac{1}{n^p}$ converges if $p>1$.) Hence by Weierstrass M – test we conclude that the given series converges uniformly and absolutely.

Example 15

Test for uniform convergence and determine the interval of convergence for

(a) $\sum_{n=1}^{\infty}\left(\frac{\cos(nx)}{n^4}\right)$ (b) $\sum_{n=1}^{\infty}\frac{x^n}{n^{3/2}}$ (c) $\sum_{n=1}^{\infty}\frac{1}{n^2+x^2}$

Solution

(a) Similar to the above Example 14. We have $f_n(x)=\frac{\cos(nx)}{n^4}$. Using Weierstrass's M – test:

$$\left|\frac{\cos(nx)}{n^4}\right|=\frac{|\cos(nx)|}{n^4}\le\frac{1}{n^4}\qquad\left[\text{Because }|\cos(nx)|\le 1\right]$$

We take our positive real numbers $M_n=\frac{1}{n^4}$. We know by the p – test that

$$\sum M_n = \sum \frac{1}{n^4} \text{ converges}$$

By Weierstrass M – test the given series $\sum_{n=1}^{\infty}\left(\frac{\cos(nx)}{n^4}\right)$ converges uniformly and absolutely for $-\infty < x < +\infty$.

(b) We first need to find the interval of convergence of the given power series $\sum_{n=1}^{\infty}\frac{x^n}{n^{3/2}}$. *How?*

By applying the ratio test we find that the interval of convergence is $-1 \le x \le 1$ or $|x| \le 1$.

For $|x| \le 1$ we have

$$\left|\frac{x^n}{n^{3/2}}\right| = \frac{|x^n|}{n^{3/2}} \le \frac{1}{n^{3/2}}$$

Let our positive real numbers $M_n = \frac{1}{n^{3/2}}$. We know by the *p* – test that

$$\sum M_n = \sum \frac{1}{n^{3/2}} \text{ converges}$$

Hence the given series $\sum_{n=1}^{\infty}\frac{x^n}{n^{3/2}}$ converges uniformly and absolutely for $-1 \le x \le 1$.

(c) The interval of convergence of the given series $\sum_{n=1}^{\infty}\frac{1}{n^2+x^2}$ is $-\infty < x < +\infty$ because by the comparison test $\frac{1}{n^2+x^2} \le \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges so $\sum_{n=1}^{\infty}\frac{1}{n^2+x^2}$ converges for all $x \in \mathbb{R}$.

We have

$$\left|\frac{1}{n^2+x^2}\right| = \frac{1}{n^2+x^2} \le \frac{1}{n^2}$$

$\sum \frac{1}{n^2}$ converges therefore by the Weierstrass M – test the given series $\sum_{n=1}^{\infty}\frac{1}{n^2+x^2}$ converges uniformly and absolutely for $-\infty < x < +\infty$.

Example 16

Show that following series is **not** uniformly convergent at $x = 0$:

$$\sum_{n=0}^{\infty} \frac{x^2}{\left(1+x^2\right)^n}$$

Solution

Suppose $\sum_{n=0}^{\infty} \frac{x^2}{\left(1+x^2\right)^n}$ is uniformly convergent at $x=0$. Writing out the given series:

$$\sum_{n=0}^{\infty} \frac{x^2}{\left(1+x^2\right)^n} = x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{\left(1+x^2\right)^2} + \frac{x^2}{\left(1+x^2\right)^3} + \frac{x^2}{\left(1+x^2\right)^4} + \cdots$$

Let $x \neq 0$ [Not zero]. This is a geometric series with the first term $a = x^2$ and common ratio $r = \frac{1}{1+x^2}$. *What is the sum of this series?*

The sum is given by the formula $S(x) = \frac{a}{1-r}$ provided $|r|<1$. Applying this formula with $a = x^2$ and $r = \frac{1}{1+x^2}$:

$$S(x) = \frac{a}{1-r} = \frac{x^2}{1-\frac{1}{1+x^2}} = \frac{x^2\left(1+x^2\right)}{1+x^2-1} = 1+x^2 \text{ where } x \neq 0$$

The limit of this function as *x* tends to zero is given by $\lim_{x\to 0}\left(S(x)\right) = \lim_{x\to 0}\left(1+x^2\right) = 1$.

However at $x=0$ the given series is equal to

$$S(0) = \sum_{n=0}^{\infty} \frac{0^2}{\left(1+0^2\right)^n} = 0^2 + \frac{0^2}{1+0^2} + \frac{0^2}{\left(1+0^2\right)^2} + \cdots = 0+0+0+0+\cdots = 0$$

Hence at $x=0$ the given series is equal to 0 but the limit of this function as $x \to 0$ is 1. Hence

$$\lim_{x\to 0}\left(S(x)\right) \neq S(0)$$

This means that $S(x)$ is discontinuous at $x=0$. The graph of $S(x)$ is shown below:

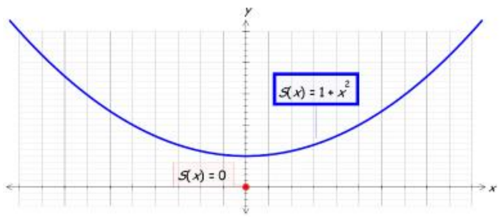

Fig 18

By the above Proposition:

Proposition (3.5). Let the sequence of functions $(f_n(x))$ converge **uniformly** to a function $f(x)$ in the domain *D*. If each $f_n(x)$ is continuous at a point *c* in the domain *D* then the limit function $f(x)$ is also continuous at point *c*.

We conclude that $S(x)$ is **not** uniformly convergent at $x=0$.

Summary

If the sequence of functions $(f_n(x))$ converges uniformly to $f(x)$ and each element $f_n(x)$ is continuous at a point $x=c$ then $f(x)$ is continuous at $x=c$.

$\sum f_n(x)$ converges uniformly and absolutely to a function $f(x)$ in the interval of convergence provided:

i) $|f_n(x)-f(x)| \leq M_n$ where $M_n > 0$

ii) (M_n) converges to 0