Section D **Properties of Limits**

By the end of this section you will be able to

- prove that the limit is unique
- prove a convergent sequence is bounded
- prove various properties of limits of a sequence

D1 **Uniqueness of the Limit**

In this section you need to know the $\varepsilon - N_0$ definition of the limit of a sequence given in the last section:

The sequence (x_n) converges to limit L if and only if for every given positive real number ε there exists a positive real number N_0 such that for all $n > N_0$ we have

(5.11) $$|x_n - L| < \varepsilon$$

In this definition we have taken N_0 to be a real number only to make the algebra of inequalities simpler. In general it is more convenient to take N_0 to be a natural number (or positive integer) because the number n in $n > N_0$ is a natural number and remember the sequence (x_n) is a function with the domain being the set of natural numbers $\mathbb{N}$. From now on the number N_0 in definition (5.11) will be a natural number. Hence the above definition becomes:

The sequence (x_n) converges to limit L if and only if for every given positive real number ε there exists a natural number N_0 such that for all $n > N_0$ we have

(5.11) $$|x_n - L| < \varepsilon$$

Remember the distance between x_n and L that is $|x_n - L| < \varepsilon$ can be made as small (ε) as we please. Note that we can find a number N_0 large enough so that the distance between x_n and L can be made **even** within any positive fraction of ε. For example we can find numbers N_0 so that we have for all $n > N_0$ the following inequalities:

$$|x_n - L| < \frac{\varepsilon}{2}, \quad |x_n - L| < \frac{\varepsilon}{3}, \quad |x_n - L| < \frac{\varepsilon}{1000} \quad \text{etc.}$$

We use this concept of the definition (5.11) to prove propositions in this section.

Note that definition (5.11) goes both ways, $\Rightarrow$ and $\Leftarrow$, because we have an 'if and only if' statement in the definition. *What does this mean in relation to the limit of a sequence?*

($\Rightarrow$). If $\lim_{n\to\infty}(x_n) = L$ then for every $\varepsilon > 0$ there is a number N_0 such that for all $n > N_0$

$$|x_n - L| < \varepsilon$$

Going the other way, $\Leftarrow$, if for every $\varepsilon > 0$ there is a number N_0 such that for all $n > N_0$

$$|x_n - L| < \varepsilon$$

then $\lim_{n\to\infty}(x_n) = L$. Hence if we establish the inequality $|x_n - L| < \varepsilon$ then we have proven $\lim_{n\to\infty}(x_n) = L$. In this section we establish inequalities like $|x_n - L| < \varepsilon$ to prove

$\lim_{n\to\infty}(x_n) = L$.

We also use the maximum function throughout this section. That is the maximum of a non-empty set S denoted $\max S$ is the largest element of the set *S*. The function max selects the largest value. For example let $S = \{1,\ 2,\ 3\}$ then $\max\{1,\ 2,\ 3\} = 3$. *What is* $\max\{1,\ 2,\ 5\}$ *equal to?*

$\max\{1,\ 2,\ 5\} = 5$. *What is* $\max\left\{\frac{1}{n}\,\middle|\, n \in \mathbb{N}\right\}$ *equal to?*

The set $\left\{\frac{1}{n}\,\middle|\, n \in \mathbb{N}\right\} = \left\{1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5}, \cdots\right\}$ therefore

$$\max\left\{\frac{1}{n}\,\middle|\, n \in \mathbb{N}\right\} = \max\left\{1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5}, \cdots\right\} = 1$$

The triangle inequality that we established in the last chapter

$$|a+b| \le |a| + |b| \qquad \text{where } a \in \mathbb{R} \text{ and } b \in \mathbb{R}$$

is used throughout this section. Make sure you know how to apply this inequality.

Another mathematical trick that we use throughout this section is we add and subtract the same quantity so keeping the equality sign. For example

$$5 = 5 - 4 + 4 \quad \text{or} \quad x = x - y + y \quad \text{or} \quad x = x + yM - yM \quad \text{etc}$$

We use this trick in several proofs below.

Proposition (5.12).

The **limit** of a real convergent sequence is **unique**.

Note. *What does this proposition mean?*

There is **only one** limit of a convergent sequence. *How do we prove this?*

We assume there are two limits, L and M, and then show they are equal, $L = M$.

How do we show $L = M$ *?*

Required to show that

$$0 \le |L - M| < \varepsilon \qquad (\$)$$

for an arbitrary $\varepsilon > 0$ then by the Proposition (4.20)[1] of the last chapter we have $L - M = 0$ which implies $L = M$. Here is the proof.

Proof. Let $\varepsilon > 0$ be given and (x_n) be a real convergent sequence which has two limits, L and M, given by

$$\lim_{n\to\infty}(x_n) = L \text{ and } \lim_{n\to\infty}(x_n) = M$$

Required to prove that $L = M$. Since we have $\lim_{n\to\infty}(x_n) = L$ therefore by the above definition (5.11) we know there is a number N_0 such that for all $n > N_0$ the following inequality holds:

$$|x_n - L| < \varepsilon_1 \qquad (*)$$

Note that the definition says we can make the distance between x_n and L as small as we please for large enough n. Therefore ε_1 is any positive real number. Similarly, from the other limit $\lim_{n\to\infty}(x_n) = M$ we have a number N_1 say such that for

all $n > N_1$ the following the inequality holds:

$$|x_n - M| < \varepsilon_2 \qquad (**)$$

The numbers N_0 and N_1 are valid for values of ε_1 and ε_2 which are determined later on.

For both inequalities to be valid we need the largest number out of N_0 and N_1 which

can be determined by the mathematical function $\max\{N_0,\ N_1\}$. Hence let $N = \max\{N_0,\ N_1\}$ then for all $n > N$ we have

[1] (4.20) $\quad 0 \le |x| < \varepsilon \quad \Rightarrow \quad x = 0$

$$
\begin{aligned}
0 \le |L-M| &= |L-x_n+x_n-M| && \text{[Subtracting and Adding } x_n\text{]} \\
&= |(L-x_n)+(x_n-M)| \\
&\le |L-x_n|+|x_n-M| && \text{[Applying the triangle inequality } |a+b|\le|a|+|b|\text{]} \\
&= |-(x_n-L)|+|x_n-M| && \text{[Rewriting } |L-x_n|=|-(x_n-L)|\text{]} \\
&= \underbrace{|x_n-L|}_{\text{Because } |-1|=1}+|x_n-M| && \text{[Because } |-(x_n-L)|=|-1||x_n-L|\text{]} \\
&< \underbrace{\varepsilon_1}_{\text{By (*)}} + \underbrace{\varepsilon_2}_{\text{By (**)}}
\end{aligned}
$$

Hence we have $0 \le |L-M| < \varepsilon_1+\varepsilon_2$. *How do we get ($) that is* $0 \le |L-M| < \varepsilon$ *?*

Let $\varepsilon_1 = \dfrac{\varepsilon}{2}$ and $\varepsilon_2 = \dfrac{\varepsilon}{2}$ then

$$0 \le |L-M| < \varepsilon_1+\varepsilon_2 = \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$$

Hence we have $0 \le |L-M| < \varepsilon$ for an arbitrary $\varepsilon > 0$. By Proposition (4.20)[2] we have $L-M=0$ which gives the required result $L=M$. Hence the **limit** of the sequence is **unique**.

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D2 **Boundedness**

In the next subsection we look at addition, subtraction, multiplication and division of limits. Before we examine these we define what is meant by a bounded sequence.

Bounded Sequence (5.13). A real sequence (x_n) is **bounded** if there exists a real number $M > 0$ such that

$$|x_n| \le M \qquad \text{for all } n \in \mathbb{N}$$

For example the sequence $x_n = \dfrac{1}{n}$ is bounded. *Why?*

Because looking at this sequence we have

$$x_n = 1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \frac{1}{6}\cdots$$

Clearly for all $n \in \mathbb{N}$ we have $|x_n| \le 1$. *Is the sequence* $x_n = n$ *bounded?*

No because if we examine $x_n = n$ we have

$$x_n = 1,\ 2,\ 3,\ 4,\ 5,\ \cdots$$

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[2] (4.20) $\qquad 0 \le |x| < \varepsilon \quad \Rightarrow \quad x = 0$

There is **no** real number M such that $|x_n| \le M$ so therefore the given sequence $x_n = n$ is **not** bounded. *Is the sequence* $x_n = 1 + (-1)^n$ *bounded?*

Yes because

$$x_n = 0,\ 2,\ 0,\ 2,\ 0,\ 2,\ \cdots$$

Since for all $n \in \mathbb{N}$ we have $|x_n| \le 2$ therefore the given sequence $x_n = 1 + (-1)^n$ is bounded.

Next we prove that a real **convergent** sequence is **bounded**. To prove this we need to apply the definition of a limit of a sequence (5.11) again.

Proposition (5.14).

A **convergent** sequence of real numbers is **bounded**.

Proof. Let (x_n) be a real sequence which converges to say the limit L, that is $\lim_{n\to\infty}(x_n) = L$. By definition (5.11) we have for any given $\varepsilon > 0$ a number N_0 such that for all $n > N_0$

$$|x_n - L| < \varepsilon \qquad (\dagger)$$

How do we show the sequence (x_n) *is bounded?*

By definition (5.13) above we need to find a real number $M > 0$ such that for all $n \in \mathbb{N}$ the sequence (x_n) satisfies $|x_n| \le M$. That is we have to show that all the $|x_n|$'s are less than or equal to M. We have

$$\begin{aligned} |x_n| &= |x_n - L + L| && [\text{Add and Subtract } L] \\ &\le |x_n - L| + |L| && [\text{By the Triangle Inequality } |a+b| \le |a| + |b|] \\ &< \underbrace{\varepsilon}_{\text{By } (\dagger)} + |L| \end{aligned}$$

Hence $|x_n| < |L| + \varepsilon$. Note this inequality is only valid for $n > N_0$ which means that

$$|x_{N_0+1}| < |L| + \varepsilon,\ |x_{N_0+2}| < |L| + \varepsilon,\ |x_{N_0+3}| < |L| + \varepsilon, \ldots$$

All the terms of the sequence after the number N_0 are less than $|L| + \varepsilon$.

Let

$$M = \max\{|x_1|,\ |x_2|,\ |x_3|,\ |x_4|,\ \cdots, |x_{N_0}|, |L| + \varepsilon\}$$

Remember the function $\max\{|x_1|,\ |x_2|,\ |x_3|,\ |x_4|,\ \cdots, |x_{N_0}|, |L| + \varepsilon\}$ selects the largest value of these elements. Therefore, all the terms of the sequence (x_n) are less than or equal to M. Hence for all $n \in \mathbb{N}$ the sequence (x_n) satisfies $|x_n| \le M$. Since we have found a real number $M > 0$ therefore by definition (5.13) we conclude that the convergent sequence (x_n) is bounded.

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Note that a sequence which is bounded may **not** be convergent. For example $x_n = (-1)^n$ is the sequence

$$x_n = -1,\ 1,\ -1,\ 1,\ -1,\ 1,\ -1,\ 1,\ \ldots$$

This sequence is *not convergent* but it is clearly *bounded* because $|x_n| \le 1$.

D3 **Arithmetic Properties of the Limit**

Proposition (5.15). Let (x_n) and (y_n) be real convergent sequences. Let

$$\lim_{n\to\infty}(x_n) = L \text{ and } \lim_{n\to\infty}(y_n) = M$$

Then

(i) $\lim_{n\to\infty}(x_n + y_n) = L + M$

(ii) $\lim_{n\to\infty}(x_n - y_n) = L - M$

(iii) $\lim_{n\to\infty}(x_n y_n) = LM$

(iv) $\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right) = \frac{L}{M}$ provided for all $n \in \mathbb{N}$ $y_n \ne 0$ and $M \ne 0$

We only prove parts (i) and (iii). You are asked to prove parts (ii) and (iv) in Exercise 5(d).

(i) *What do we need to prove?*

Required to show that for large enough n we can make the distance between $(x_n + y_n)$ and $(L+M)$ as small as we please, that is $|(x_n + y_n) - (L+M)| < \varepsilon$ where $\varepsilon > 0$ is arbitrary. Then by (5.11) we have $\lim_{n\to\infty}(x_n + y_n) = L + M$. Here is the proof.

Proof. Let $\varepsilon > 0$ be an arbitrary real number. Since we are given $\lim_{n\to\infty}(x_n) = L$ therefore by definition (5.11) there is a number N_0 such that for all $n > N_0$ we have

$$|x_n - L| < \varepsilon_1 \qquad (*)$$

Similarly from $\lim_{n\to\infty}(y_n) = M$ there is a number N_1 say such that for all $n > N_1$ we have

$$|y_n - M| < \varepsilon_2 \qquad (**)$$

The numbers ε_1 and ε_2 are determined later. Both these inequalities, (*) and (**), are valid for numbers greater than N_0 and N_1. Let $N = \max\{N_0,\ N_1\}$ the largest value out of N_0 and N_1. Consider the term $|(x_n + y_n) - (L+M)|$, then for all $n > N$ we have

$$\left|(x_n + y_n) - (L+M)\right| = \left|x_n - L + y_n - M\right| \qquad [\text{Opening Brackets}]$$

$$\leq \left|x_n - L\right| + \left|y_n - M\right| \qquad \begin{bmatrix} \text{By the Triangle Inequality} \\ |a+b| \leq |a| + |b| \end{bmatrix}$$

$$< \underbrace{\varepsilon_1}_{\text{By (*)}} + \underbrace{\varepsilon_2}_{\text{By (**)}}$$

We have

$$\left|(x_n + y_n) - (L+M)\right| < \varepsilon_1 + \varepsilon_2 \qquad (\$)$$

Required to prove $\left|(x_n + y_n) - (L+M)\right| < \varepsilon$. *How?*

By letting $\varepsilon_1 = \frac{\varepsilon}{2}$ and $\varepsilon_2 = \frac{\varepsilon}{2}$ then from ($)

$$\left|(x_n + y_n) - (L+M)\right| < \varepsilon_1 + \varepsilon_2$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence $\left|(x_n + y_n) - (L+M)\right| < \varepsilon$ for all $n > N$ therefore by definition (5.11) we have $\lim_{n\to\infty}(x_n + y_n) = L + M$.

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(ii) See Exercise 5(d)

(iii) Since we want $\lim_{n\to\infty}(x_n y_n) = LM$ therefore it is enough to prove that $\left|x_n y_n - LM\right| < \varepsilon$ for any arbitrary $\varepsilon > 0$ and large enough n. This means we can make the distance between $x_n y_n$ and LM as small as we please for large enough n.

Here is the proof.

Proof. Let $\varepsilon > 0$ be arbitrary. Since we are given $\lim_{n\to\infty}(x_n) = L$ therefore by definition (5.11) there is a number N_0 such that for all $n > N_0$ we have

$$\left|x_n - L\right| < \varepsilon_1 \qquad (\dagger)$$

Similarly from $\lim_{n\to\infty}(y_n) = M$ there is a number N_1 say such that for all $n > N_1$ we have

$$\left|y_n - M\right| < \varepsilon_2 \qquad (\dagger\dagger)$$

Values of ε_1 and ε_2 are determined later.

Since the sequence (x_n) is convergent therefore by proposition (5.14) it is bounded which means there is a real number $K > 0$ such that for all $n \in \mathbb{N}$

$$\left|x_n\right| \leq K \qquad (*)$$

Let $N = \max\{N_0,\ N_1\}$ so that for all $n > N$ both inequalities (†) and (††) are valid. Consider the term $\left|x_n y_n - LM\right|$, then for all $n > N$ we have

$$\begin{aligned}
|x_n y_n - LM| &= |x_n y_n - x_n M + x_n M - LM| && [\text{Add and Subtract } x_n M] \\
&\le |x_n y_n - x_n M| + |x_n M - LM| && [\text{By } |a+b| \le |a| + |b|] \\
&= |x_n||y_n - M| + |M||x_n - L| && [\text{Taking Out Common Factors}] \\
&< |x_n| \underbrace{\varepsilon_2}_{\text{By } (\dagger\dagger)} + |M| \underbrace{\varepsilon_1}_{\text{By } (\dagger)} \\
&\le \underbrace{K}_{\text{By } (*)} \varepsilon_2 + |M| \varepsilon_1
\end{aligned}$$

Hence, we have $|x_n y_n - LM| < K\varepsilon_2 + |M|\varepsilon_1$ but we need to show $|x_n y_n - LM| < \varepsilon$.

How?

Let $\varepsilon_2 = \dfrac{\varepsilon}{2K}$ and $\varepsilon_1 = \dfrac{\varepsilon}{2|M|}$ then we have

$$\begin{aligned}
|x_n y_n - LM| &< K\varepsilon_2 + |M|\varepsilon_1 \\
&= K\frac{\varepsilon}{2K} + |M|\frac{\varepsilon}{2|M|} && \left[\text{Substituting } \varepsilon_1 = \frac{\varepsilon}{2|M|} \text{ and } \varepsilon_2 = \frac{\varepsilon}{2K}\right] \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon && [\text{Cancelling Out the } K\text{'s and } |M|\text{'s}]
\end{aligned}$$

Hence we have $|x_n y_n - LM| < \varepsilon$ for all $n > N$. By the definition of the limit of a sequence (5.11) we have the required result, $\lim_{n\to\infty}(x_n y_n) = LM$.

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(iv) See Exercise 5(d)

Proposition (5.16).

Let (x_n) be a real convergent sequence such that $\lim_{n\to\infty}(x_n) = L$. If for all $n \in \mathbb{N}$ we have $x_n \ge 0$ then $\lim_{n\to\infty}(x_n) = L \ge 0$.

Note. *What does this proposition mean?*

It says if all the terms of the convergent sequence (x_n) are positive or zero then the limiting value of this sequence will also be positive or zero.

We use proof by contradiction. *What is our supposition going to be?*

Suppose the limit $L < 0$ and then by logical deduction we should arrive at some contradiction. Here is the proof.

Proof. Suppose $L < 0$. Since $\lim_{n\to\infty}(x_n) = L$ therefore by definition (5.11) we have for every $\varepsilon > 0$ a natural number N_0 such that for all $n > N_0$

$$|x_n - L| < \varepsilon$$

What does this $|x_n - L| < \varepsilon$ mean?

Remember by our definition of the modulus function $|x_n - L| < \varepsilon$ means

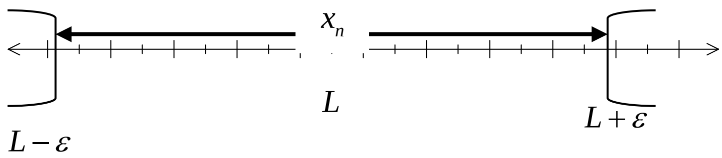

Fig 1

$$L-\varepsilon < x_n < L+\varepsilon$$

Hence x_n lies between $L-\varepsilon$ and $L+\varepsilon$. This inequality only holds for $n > N_0$. Therefore with $n = N_0 + 1$ we have

$$L-\varepsilon < x_{N_0+1} < L+\varepsilon$$

Considering **only** the right hand inequality

$$x_{N_0+1} < L+\varepsilon$$

Since this is valid for any $\varepsilon > 0$ we can take $\varepsilon = -L$ because $L < 0$ so therefore

$$x_{N_0+1} < L - L = 0$$

Hence $x_{N_0+1} < 0$. This is a contradiction. *Why?*

Because in the proposition we are given 'for all $n \in \mathbb{N}$ we have $x_n \geq 0$' but we have

found a number $n = N_0 + 1$ such that $x_{N_0+1} < 0$. We **cannot** have both $x_{N_0+1} \geq 0$ and $x_{N_0+1} < 0$.

Hence our supposition that $L < 0$ must be false, so $L \geq 0$ which is the required result.

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Proposition (5.17). Let both (x_n) and (y_n) be real convergent sequences. If for all $n \in \mathbb{N}$ we have the inequality

$$x_n \geq y_n$$

Then $\lim_{n\to\infty}(x_n) \geq \lim_{n\to\infty}(y_n)$.

Proof. Since for all $n \in \mathbb{N}$ we have $x_n \geq y_n$ therefore

$$x_n - y_n \geq 0 \quad \text{for all } n \in \mathbb{N}$$

By the above proposition (5.16) we have

$$\lim_{n\to\infty}(x_n - y_n) \geq 0 \qquad (*)$$

By proposition (5.15) part (ii) we have

$$\lim_{n\to\infty}(x_n - y_n) = \lim_{n\to\infty}(x_n) - \lim_{n\to\infty}(y_n)$$

Substituting this into (*) yields

$$\lim_{n\to\infty}(x_n) - \lim_{n\to\infty}(y_n) \ge 0$$

$$\lim_{n\to\infty}(x_n) \ge \lim_{n\to\infty}(y_n)$$

Hence, we have our result.

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SUMMARY

(5.12) The **limit** of a sequence is **unique**.

(5.13) A sequence (x_n) is **bounded** if there exists a real number $M > 0$ such that

$$|x_n| \le M \quad \text{for all } n \in \mathbb{N}$$

(5.14) A **convergent** sequence is **bounded**.

(5.15) If $\lim_{n\to\infty}(x_n) = L$ and $\lim_{n\to\infty}(y_n) = M$ then

(i) and (ii) $\lim_{n\to\infty}(x_n \pm y_n) = L \pm M$

(iii) $\lim_{n\to\infty}(x_n y_n) = LM$

(iv) $\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right) = \frac{L}{M}$ provided $\forall n \in \mathbb{N}$ $y_n \ne 0$ and $M \ne 0$

(5.16) If for all $n \in \mathbb{N}$ the sequence $x_n \ge 0$ and is convergent then $\lim_{n\to\infty}(x_n) \ge 0$.