

Exercises 3.1

Question 9

$$3^{100} \equiv r \pmod{10}$$

Soln: We have

$$3^2 \equiv 9 \equiv \underline{-1} \pmod{10}$$

$$100 = 2 \times 50$$

$$3^{100} \equiv 3^{2 \times 50}$$

$$\equiv (3^2)^{50}$$

$$\equiv (-1)^{50} \equiv \underline{\underline{1}} \pmod{10}$$

(d) $4^{100} \equiv r \pmod{10}$

Soln: Note that $4 = 2^2$.

$$4^{100} \equiv (2^2)^{100}$$

$$\equiv (2^{100})^2$$

(*)

$$2^5 \equiv 32 \equiv 2 \pmod{10}$$

$$2^{100} \equiv 2^{5 \times 20}$$

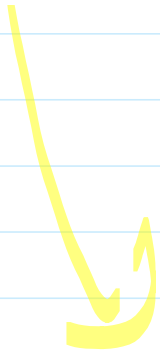
$$\equiv (2^5)^{20}$$

$$\equiv 2^{20}$$

$$\equiv 2^{5 \times 4}$$

$$\equiv (2^5)^4$$

$$\begin{aligned} &= 2^4 \\ &\equiv (2^5)^4 \\ &\equiv 2^4 \equiv 16 \equiv 6 \pmod{10} \end{aligned}$$



$$\begin{aligned} 4^{100} &\equiv (2^{100})^2 \equiv 6^2 \\ &\equiv 36 \\ &\equiv 6 \pmod{10}. \end{aligned}$$

Exercises 3.1 Question 10 *

Last two digits 2014^{2014}

Solⁿ: Work with modulo 100. Note

$$2014 \equiv 14 \pmod{100}$$

$$\underline{2014}^{2014} \equiv \underline{14}^{2014} \pmod{100} \quad (*)$$

Evaluating powers of 14:

$$14^2 \equiv 196 \equiv 96 \equiv \underline{-4} \pmod{100}$$

Substituting into (*) gives

$$\begin{aligned} \underline{2014}^{2014} &\equiv 14^{2 \times 1007} \\ &\equiv (14^2)^{1007} \\ &\equiv (-4)^{1007} \\ &\equiv (-1 \times 4)^{1007} \\ &\equiv (-1)^{1007} \cdot 4^{1007} \end{aligned}$$

odd

$$\equiv -1 \cdot \underline{4^{1007}} \pmod{100} \quad (**)$$

Evaluating powers of 4:

$$\underline{4^{11}} \equiv \underline{4} \pmod{100} \quad (***)$$

By the D.A.

$$1007 = (91 \times 11) + 6$$

Therefore

$$\begin{aligned} \underline{4^{1007}} &\equiv 4^{(91 \times 11) + 6} \\ &\equiv (4^{11})^{91} \times 4^6 \\ &= (11) \cdot 91 \times 4^6 \end{aligned}$$

$$\equiv (4)^{a_1} \times 4^6$$

$$\equiv 4^{(8 \times 11) + 3} \times 4^6$$

$$\equiv (4^{11})^8 \times 4^3 \times 4^6$$

$$\equiv 4^8 \times 4^3 \times 4^6$$

$$\equiv 4^{8+3} \times 4^6$$

$$\equiv (4^{11}) \times 4^6$$

$$\equiv 4 \times 4^6$$

$$\equiv 4^7 \equiv 84 \equiv -16 \pmod{100}$$

$$2014^{2014} \equiv (-1) \times (-16)$$

$$\equiv 16 \pmod{100}$$

2014²⁰¹⁴ are 16.

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The last digit of a square can only be 0, 1, 4, 5, 6 and 9.

Proof: Let n be any integer, then D.A.

$$n = 10q + r \quad 0 \leq r < 10$$

Squaring this yields

$$n^2 = (10q + r)^2$$

$$= \underbrace{(10q)^2 + 2(10q)r + r^2}_{10m \equiv 0 \pmod{10}}$$

$$\equiv 0 + r^2 \equiv r^2 \pmod{10}.$$

r can only be 0, 1, 2, 3, ..., 9

$$n^2 \equiv r^2 \equiv 0^2, 1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2, 9^2$$

$$\equiv 0, 1, 4, 9, 16, 25, 36, 49, 64, 81$$

$$\equiv 0, 1, 4, 9, 6, 5, 6, 9, 4, 1 \pmod{10}$$

Hence

$$n^2 \equiv 0, 1, 4, 5, 6, 9 \pmod{10}$$