Exercises 3.1

Question 9

$$3^{100} \equiv r \overset{1}{=} (\text{mod } 10)$$

Sol^n: We have

$$3^2 \equiv 9 \equiv -1 \ (\text{mod } 10)$$

$$100 = 2 \times 50$$

$$3^{100} \equiv 3^{2\times 50}$$

$$\equiv (3^2)^{50}$$

$$\equiv (-1)^{50} \equiv 1 \ (\text{mod } 10)$$

(d) $4^{100} \equiv r \ (\text{mod } 10)$

Sol^n: Note that $4 = 2^2$.

$$4^{100} \equiv (2^2)^{100}$$

$$\equiv (2^{100})^2 \qquad (*)$$

$$2^5 \equiv 32 \equiv 2 \ (\text{mod } 10)$$

$$2^{100} \equiv 2^{5\times 20}$$

$$\equiv (2^5)^{20}$$

$$\equiv 2^{20}$$

$$\equiv 2^{5\times 4}$$

$$\equiv (2^5)^4$$

$\equiv 2$

$\equiv (2^5)^4$

$\equiv 2^4 \equiv 16 \equiv 6 \pmod{10}$

$4^{100} \equiv (2^{100})^2 \equiv 6^2$

$\equiv 36$

$\equiv 6 \pmod{10}.$

Exercises 3.1 Question 10 *

Last two digits 2014^{2014}

Soln: Work with modulo 100. Note

$$2014 \equiv 14 \pmod{100}$$

$$2014^{2014} \equiv 14^{2014} \pmod{100} \quad (\dagger)$$

Evaluating powers of 14:

$$14^2 \equiv 196 \equiv 96 \equiv -4 \pmod{100}$$

Substituting into $(\dagger)$ gives

$$2014^{2014} \equiv 14^{2\times 1007}$$
$$\equiv (14^2)^{1007}$$
$$\equiv (-4)^{1007}$$
$$\equiv (-1 \times 4)^{1007}$$

odd

$$\equiv (-1)^{1007}\, 4^{1007}$$
$$\equiv -1 \cdot 4^{1007} \pmod{100} \quad (*)$$

Evaluating powers of 4:

$$4^{11} \equiv 4 \pmod{100} \quad (**)$$

By the D.A. $\quad 1007 = (91 \times 11) + 6$

Therefore

$$4^{1007} \equiv 4^{(91\times 11)+6}$$
$$\equiv (4^{11})^{91} \times 4^{6}$$
$$\equiv (4)^{91} \times 4^{6}$$

$$\equiv (4)^{91} \times 4^{6}$$

$$\equiv 4^{(8\times 11)+3} \times 4^{6}$$

$$\equiv (4^{11})^{8} \times 4^{3} \times 4^{6}$$

$$\equiv 4^{8} \times 4^{3} \times 4^{6}$$

$$\equiv 4^{8+3} \times 4^{6}$$

$$\equiv 4^{11} \times 4^{6}$$

$$\equiv 4 \times 4^{6}$$

$$\equiv 4^{7} \equiv 84 \equiv -16 \pmod{100}$$

$$2014^{2014} \equiv (-1)\times(-16)$$

$$\equiv 16 \pmod{100}$$

2014^{2014} are 16.

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The last digit of a square can only be
0, 1, 4, 5, 6 and 9.

Proof: Let n be any integer, then D.A.

$$n = 10q + r \qquad 0 \le r < 10$$

Squaring this yields

$$n^2 = (10q + r)^2$$

$$= \underbrace{(10q)^2 + 2(10q)r}_{10m \equiv 0 \pmod{10}} + r^2$$

$$\equiv 0 + r^2 \equiv r^2 \pmod{10}.$$

r can only be $0, 1, 2, 3, \ldots, 9$

$$n^2 \equiv r^2 \equiv 0^2, 1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2, 9^2$$

$$\equiv 0, 1, 4, 9, 16, 25, 36, 49, 64, 81$$

$$\equiv 0, 1, 4, 9, 6, 5, 6, 9, 4, 1 \pmod{10}$$

Hence

$$n^2 \equiv 0, 1, 4, 5, 6, 9 \pmod{10}$$