Complete Solutions to Exercises I.5

1. In each case we solve the given equation and we obtain the following results:

(a) $A=\left\{-\frac{1}{2}\right\}$ (b) $B=\left\{1,\ 2\right\}$ (c) $C=\left\{1\right\}$

(d) $D=\left\{-1,\ 5\right\}$ (e) $E=\left\{-3,\ 3\right\}$

(f) $F=\left\{2,\ 3,\ 5,\ 7\right\}$ (these are the prime numbers less than 10.)

2. This time be careful because you need to look at the universal set.

(a) The solution to the give quadratic equation

$$\left(x-1\right)\left(x+3\right)=0 \ \Rightarrow\ x=1,\ x=-3\,.$$

However our universal set is that natural numbers $\mathbb{N}$ so the only member of the given set A is 1, that is $A=\left\{1\right\}$.

(b) This time solving $2x+1=0 \ \Rightarrow\ x=-\frac{1}{2}$. But we are given

$$B=\left\{x\in\mathbb{N}:2x+1=0\right\}.$$

Clearly $-\frac{1}{2}$ is *not* a natural number so $B=\varnothing$.

(c) We are given the set $C=\left\{x\in\mathbb{Z}:\left(x+5\right)\left(3x-1\right)=0\right\}$ and solving the quadratic in this set yields $x=-5,\ x=\frac{1}{3}$. Only -5 is an integer so

$$C=\left\{-5\right\}.$$

(d) This time we have the same quadratic in part (c) so we have the same solution $x=-5,\ x=\frac{1}{3}$ but universal set is the rationals $\mathbb{Q}$ so

$$D=\left\{\frac{1}{3},\ -5\right\}$$

(e) The given linear equation $x-\pi=0 \ \Rightarrow\ x=\pi$. We haven't shown this but π is *not* a rational number so $E=\varnothing$.

(f) We have the same equation as part (e) but the universal set is $\mathbb{R}$ and π is a real number so $F=\left\{\pi\right\}$.

3. Using the set notation, we have the following:

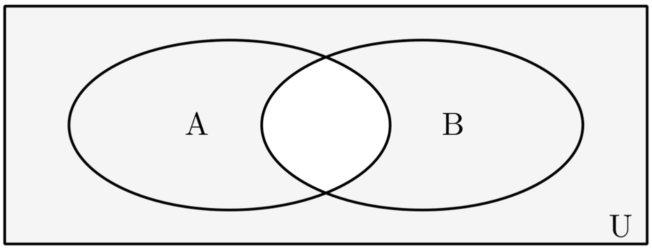
(a) $\left\{x\in\mathbb{R}:x<0\right\}$ (b) $\left\{x\in\mathbb{Z}:x>0\right\}$ or $\mathbb{N}$ (c) $\left\{x\in\mathbb{R}:0<x<2\right\}$

(d) $\{x \in \mathbb{Q} : x < 1\}$ (e) $\{10n : n \in \mathbb{N}\}$

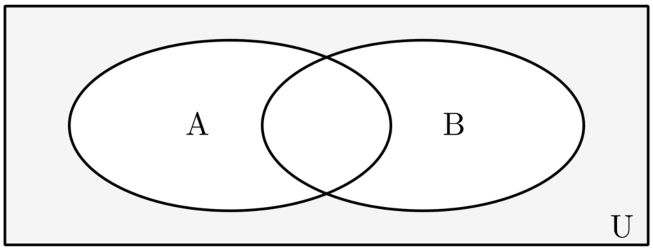
4. (a) See Fig. 12.

(b) Same as Fig. 12 with the set labelled B instead of A.

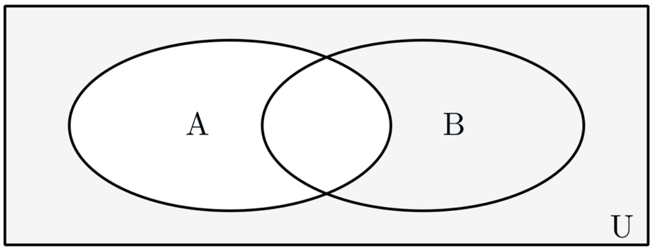
(c) The Venn diagram for $(A \cap B)^c$ is shaded below:



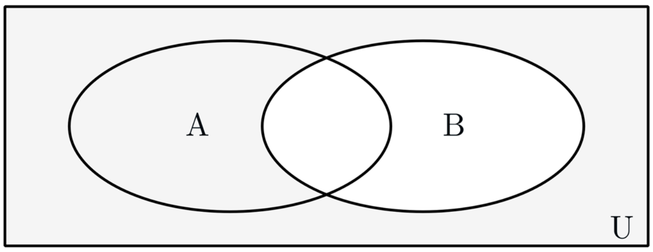
(d) The Venn diagram for $(A \cup B)^c$ is shaded:



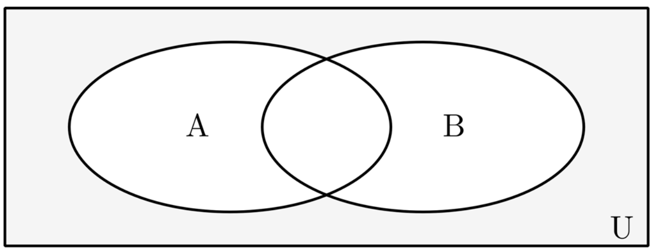
(e) The Venn diagram for $A^c \cap B^c$ means we first shade outside of the set A:



Now we shade outside of B:



The shading overlapping of the last two Venn diagrams gives us everything outside of $A \cup B$ which is:

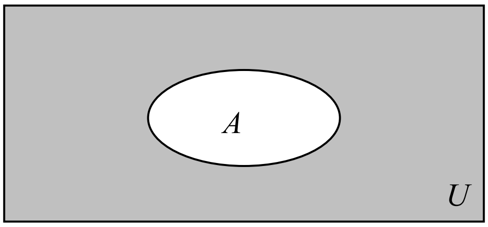


Notice that according to these Venn diagrams we have

$$(A \cup B)^c = A^c \cap B^c.$$

It can be shown that this result always holds, that is $(A \cup B)^c = A^c \cap B^c$.

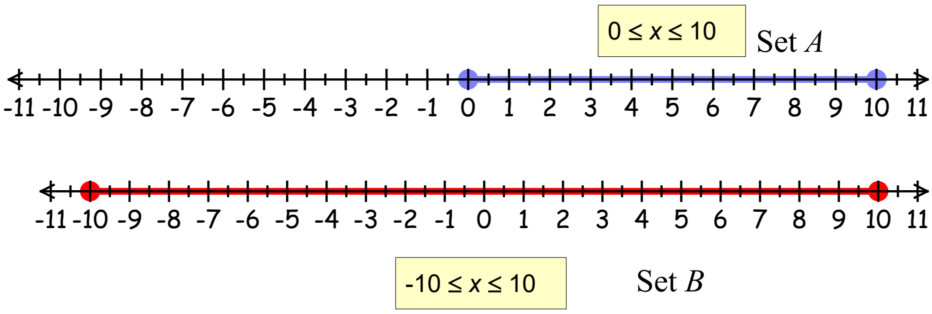
5. From Fig. 12 we have



A^c

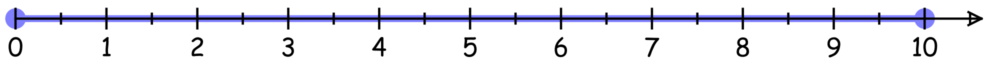
Now for $(A^c)^c$ we only shade the blank part in this Venn diagram which gives us the set A. Hence $(A^c)^c = A$.

6. (a) We can draw the given sets $A = \{x \in \mathbb{R} : 0 \le x \le 10\}$ and $B = \{x \in \mathbb{R} : -10 \le x \le 10\}$ on the number line as follows:



Clearly the set A is a subset of B, that is $A \subseteq B$.

(b) Again, drawing the given sets $A = \{x \in \mathbb{Z} : 0 \le x \le 10\}$ and $B = \{x \in \mathbb{R} : 0 \le x \le 10\}$ we have



The set A is the set of all the integers (whole numbers) between 0 to 10, that is $A = \{0, 1, 2, \cdots, 10\}$ whilst the set B is all the real numbers between 0 to 10, that is all the line in the above diagram. Again, the set A is a subset of the set B, that is $A \subseteq B$.

(c) This is similar to the sets in part (b) but we have switched sets A and B. Therefore, we have $B \subseteq A$.

(d) *What does the set notation* $A = \{x \in \mathbb{N} : -10 \le x \le 10\}$ *mean?*

It means that x is a natural number between -10 to 10. *What numbers does this set include?*

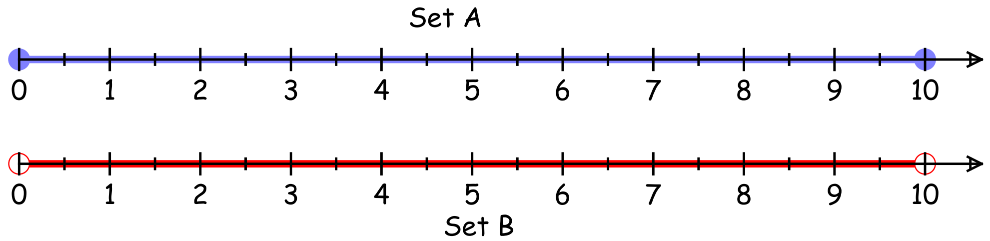
Remember natural numbers are whole numbers greater than or equal to 1, therefore $A = \{1, 2, 3, \cdots, 10\}$. *What does* $B = \{x \in \mathbb{Z} : 1 \le x \le 10\}$ *represent?*

It is the same as the set A because we have integers between 1 to 10, that is

$$B = \{1, 2, 3, \cdots, 10\}.$$

Hence we have $A \subseteq B$ and $B \subseteq A$ which means we have $A = B$.

(e) Both the given sets are the same but one of them includes the end points:



Clearly $B \subseteq A$ because the set B does *not* include the end points 0 and 10.

7. Before we evaluate which of the given sets are subsets, we need to write down the members of each given set:

$$\varnothing, \quad A = \{2, 3, 5\}, \quad B = \{x \in \mathbb{Z} : x^2 - 4 = 0\}$$
$$C = \{x \in \mathbb{N} : x \text{ is prime and less than } 10\}, \quad D = \{x \in \mathbb{Z} : 0 \le x \le 10\}$$

Then we check for subsets.

(a) For the given sets $\varnothing$ and A we have $\varnothing \subseteq A$ because the empty set $\varnothing$ is a subset of every set.

(b) *Is* A *subset of* A*?*

Yes we have $A \subseteq A$. [Every set is a subset of itself].

(c) We need to examine the given sets $A = \{2, 3, 5\}$ and $C = \{2, 3, 5, 7\}$.

Clearly all the elements of the set A which are 2, 3 and 5 are in the set C therefore A is a subset of C, that is $A \subseteq C$.

(d) Similarly, we have $C=\{2,\ 3,\ 5,\ 7\}$ and $D=\{0,\ 1,\ 2,\ 3,\ \cdots,\ 10\}$.

Again all the elements of the set C which are 2, 3, 5 and 7 are in the set D therefore C is a subset of D, that is $C\subseteq D$.

(e) *Is the set B subset of the set C?*

We have $B=\{-2,\ 2\}$ and $C=\{2,\ 3,\ 5,\ 7\}$ but the member -2 in the set B is *not* in the set C therefore $B\not\subseteq C$. *What does this notation mean?*

The set B is not a subset of the set C.

8. We need to write out the elements of each of the given sets.

$$A=\{x\in\mathbb{Z}:0<x<5\},\ \ B=\{x\in\mathbb{N}:x\text{ is an even number}\}$$
$$C=\{x\in\mathbb{N}:x\ \text{ is a multiple of }2\}$$
$$D=\{x\in\mathbb{R}:x\neq x\},\ \ E=\{x\in\mathbb{N}:x^3\}$$
$$F=\{x\in\mathbb{Z}:0<x<2\}$$

Note that the set D is the empty set because there is no real number x such that $x\neq x$.

(a) We have $A=\{1,\ 2,\ 3,\ 4\}$ and $B=\{2,\ 4,\ 6,\ 8,\ \cdots\}$ therefore $A\not\subseteq B$ because the elements 1, 3, 5, … are *not* in the set B.

(b) Similarly, we have $A=\{1,\ 2,\ 3,\ 4\}$ and $C=\{2,\ 4,\ 6,\ 8,\ \cdots\}$ therefore $A\not\subseteq C$.

(c) We have $B=\{2,\ 4,\ 6,\ 8,\ \cdots\}$ and $C=\{2,\ 4,\ 6,\ 8,\ \cdots\}$ which means we have $B\subseteq C$.

(d) B and C are the same sets as in part (c), that is

$$B=\{2,\ 4,\ 6,\ 8,\ \cdots\}\text{ and }C=\{2,\ 4,\ 6,\ 8,\ \cdots\}$$

therefore $C\subseteq B$. In fact, $B=C$.

(e) We have $D=\varnothing$ and $A=\{1,\ 2,\ 3,\ 4\}$ therefore the set A cannot be a subset of the empty set $\varnothing$ which means we have $A\not\subseteq D$.

(f) Because $D=\varnothing$ and the empty set is a subset of every set therefore $D\subseteq A$.

(g) We have the sets $E=\{1,\ 8,\ 27,\ 64,\ \cdots\}$ and $F=\{1\}$. Thus the set $E=\{1,\ 8,\ 27,\ 64,\ \cdots\}$ *cannot* be a subset of the set $F=\{1\}$. We have $E\not\subseteq F$.

(h) As part (g) we have $E = \{1,\ 8,\ 27,\ 64,\ \cdots\}$ and $F = \{1\}$ and since the member 1 is in the set $E = \{1,\ 8,\ 27,\ 64,\ \cdots\}$ therefore $F \subseteq E$.

9. *What does the term cardinality mean?*

Cardinality is the number of elements in the set and is denoted by $|A|$.

(a) Since $\varnothing$ denotes the empty set which means it has *no* elements therefore

$$|\varnothing| = 0.$$

(b) We are given the set $A = \{a,\ b,\ c\}$ therefore $|A| = 3$ because the set has 3 members.

(c) We are given $A = \{x \in \mathbb{Z} : 3x^2 - x = 0\}$ and we need to find the elements of this set A. Solving the given quadratic

$$\begin{aligned} 3x^2 - x &= 0 \\ x(3x-1) &= 0 \qquad [\text{Factorising}] \\ x = 0,\ x &= \frac{1}{3} \end{aligned}$$

Since $x \in \mathbb{Z}$ therefore members of this set A can only be integers (whole numbers) which means only 0 is a member. Thus $A = \{0\}$. *What is the cardinality of this set?*

Since the set is a singleton (only one element) therefore the cardinality $|A| = 1$.

(d) *What are the elements of the given set* $A = \{x \in \mathbb{R} : x = x + 1\}$*?*

For any real number x we have $x \neq x + 1$ therefore there are no x values which satisfy the equation $x = x + 1$. Hence the set A is empty, that is $A = \varnothing$ and so $|A| = 0$.

10. We are given the sets

$$A = \{1,\ 2,\ 3,\ 4,\ 5\} \text{ and } B = \{x \in \mathbb{N} : x \text{ is a prime number } \leq 5\}.$$

What are the elements of the set B?

$$B = \{x \in \mathbb{N} : x \text{ is a prime number } \leq 5\} = \{2, 3, 5\}.$$

How can we show $A \not\subseteq B$ *?*

Recall by Definition (I.22) we have $A \subseteq B$ if every element of set A is also in the set B. In this case we have $1 \in A$ but $1 \notin B$ therefore $A \not\subseteq B$.

How do we show $B \subseteq A$?

Again, by using the definition (I.22) we show that every element of the set B is also in the set A. We have $B = \left\{2,\ 3,\ 5\right\}$ and all three elements 2, 3 and 5 are in the set $A = \left\{1,\ 2,\ 3,\ 4,\ 5\right\}$ therefore $B \subseteq A$.

11. We have $A = \left\{1,\ 3\right\}$, $B = \left\{1,\ 3,\ 3,\ 1\right\}$ and $C = \left\{1,\ 3,\ \frac{3}{1},\ \frac{\pi}{\pi}\right\}$.

Remember from main text that a set such as $\left\{a,\ b\right\}$ is the same as $\left\{a,\ b,\ a,\ b\right\}$. Here we have

$$B = \left\{1,\ 3,\ 3,\ 1\right\} = \left\{1,\ 3\right\}$$

$$C = \left\{1,\ 3,\ \frac{3}{1},\ \frac{\pi}{\pi}\right\} = \left\{1,\ 3,\ 3,\ 1\right\} = \left\{1,\ 3\right\}$$

Thus, we have $A = B = C = \left\{1,\ 3\right\}$.

12. *Proof.* (By Contradiction).

Suppose there is an integer n such that

$$\sum_{m=1}^{n} m = 1 + 2 + 3 + 4 + \ldots + n \neq \frac{n\left(n+1\right)}{2} \qquad (*)$$

Consider the set S given by

$$S = \left\{n \in \mathbb{N} : 1 + 2 + \cdots + n \neq \frac{n\left(n+1\right)}{2}\right\}$$

By (*) the set S is non – empty. By the Well Ordering Principle (WOP) there is a least element, say m, which is a member of the set S. Note that $m > 1$ because for $n = 1$ we have our given result. Clearly $m - 1 \notin S$ because m is the least positive integer in S. This implies the given proposition is true of $m - 1$:

$$1 + 2 + 3 + \ldots + \left(m-1\right) = \frac{\left(m-1\right)\left(m-1+1\right)}{2} = \frac{m\left(m-1\right)}{2} \qquad (**)$$

Using this (**) to find the sum of the first m terms gives

$$1 + 2 + 3 + \ldots + \left(m-1\right) + m = \frac{m\left(m-1\right)}{2} + m$$

$$= \frac{m^2 - m + 2m}{2} = \frac{m^2 + m}{2} = \frac{m\left(m+1\right)}{2}$$

This implies the given proposition is true when $n = m$. Therefore, m *cannot* be the least positive integer where the given proposition is false. This is a contradiction so,

there is *no* least integer where the given proposition is false.

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13. We need to prove

Principle of Mathematical Induction (I.15)
For each natural number n, let $P(n)$ be a proposition about n. If $P(n)$ satisfies: 1) $P(1)$ is true, 2) For an arbitrary k, $P(k)$ is true implies $P(k+1)$ is true. Then for *all* natural numbers, n, we have $P(n)$ is true.

By using the WOP.

Proof.

Suppose the result is *not* true for *all* the natural numbers. There is n such that $P(n)$ is false. Let S be the subset of natural numbers where $P(n)$ is false. Then S is non – empty. By the Well Ordering Principle:

(I.24) Every non-empty subset of positive integers has a *least* element.

The set S has a lease element. Let ℓ be this least element. Then $1 \notin S$ therefore $\ell > 1$ which implies that $\ell - 1 > 0$ is a natural number. However $\ell - 1$ *cannot* be in S because $\ell - 1 < \ell$ and ℓ is the least element of S. With $\ell - 1 \notin S$ which implies that $P(n)$ is true for $n = \ell - 1$. Applying step 2 of the induction principle (I.15) on $n = \ell - 1$ gives that

$$n + 1 = \ell - 1 + 1 = \ell$$

Therefore the $P(\ell)$ is true but this is impossible because ℓ is in S which implies that $P(\ell)$ is false. We have a contradiction, that is S is an empty set and the result is true for all natural numbers. This completes our proof.

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