Complete Solutions to Exercises I.2

1. We can construct the truth tables to show that the given propositions are tautologies.

(a)

P	$\neg P$	$(\neg P) \vee P$
T	F	T
F	T	T

Hence $(\neg P) \vee P$ is a tautology.

(b) We have

Col 1	Col 2	Col 3	Col 4	Col 5	Col 6	Col 7	Col 8	Col 9
P	Q	R	$P \Rightarrow Q$	$P \Rightarrow R$	(Col 4) $\wedge$ (Col 5)	$Q \wedge R$	$P \Rightarrow (Q \wedge R)$	(Col 6) $\Rightarrow$ (Col 8)
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	F	T	F	F	F	T
F	T	T	T	T	T	T	T	T
T	F	F	F	F	F	F	F	T
F	T	F	T	T	T	F	T	T
F	F	T	T	T	T	F	T	T
F	F	F	T	T	T	F	T	T

By looking at the right hand column we can say the following is a tautology:

$$\left[(P \Rightarrow Q) \wedge (P \Rightarrow R)\right] \Rightarrow \left[P \Rightarrow (Q \wedge R)\right].$$

(c) Similarly we have

Col 1	Col 2	Col 3	Col 4	Col 5	Col 6	Col 7	Col 8	Col 9
P	Q	R	$P \Rightarrow Q$	$R \Rightarrow Q$	(Col 4) $\wedge$ (Col 5)	$P \vee R$	$(P \vee R) \Rightarrow Q$	(Col 6) $\Rightarrow$ (Col 8)
T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T	T
T	F	T	F	F	F	T	F	T

F	T	T	T	T	T	T	T	T
T	F	F	F	T	F	T	F	T
F	T	F	T	T	T	F	T	T
F	F	T	T	F	F	T	F	T
F	F	F	T	T	T	F	T	T

Hence

$$\left[\left(P \Rightarrow Q\right) \wedge \left(R \Rightarrow Q\right)\right] \Rightarrow \left[\left(P \vee Q\right) \Rightarrow Q\right] \text{ is a tautology.}$$

(d) To show $\left[\left(P \Rightarrow Q\right) \wedge \left(\neg Q\right)\right] \Rightarrow \left(\neg P\right)$ is a tautology we have to construct the truth table:

P	Q	$P \Rightarrow Q$	$\neg Q$	$\left(P \Rightarrow Q\right) \wedge \left(\neg Q\right)$	$\neg P$	$\left[\left(P \Rightarrow Q\right) \wedge \left(\neg Q\right)\right] \Rightarrow \left(\neg P\right)$
T	T	T	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

The truth values in the right-hand column are *all* true therefore

$$\left[\left(P \Rightarrow Q\right) \wedge \left(\neg Q\right)\right] \Rightarrow \left(\neg P\right)$$

is a tautology.

2. (a) *Proof.* We assume m and n are even. By Definition (I.1) they can be written as

$$n = 2a \quad \text{and} \quad m = 2b$$

where a and b are integers. Consider their addition $n + m$:

$$\begin{aligned} n + m &= 2a + 2b \\ &= 2\left(a + b\right) \qquad \left[\text{Factorizing}\right] \end{aligned}$$

We have $n + m$ is of the form 2(An Integer). By applying Definition (I.1) in the $\Leftarrow$ direction we conclude that $n + m$ is even.

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(b) *Proof.* We assume m and n are even. By Definition (I.1) they can be written as

$$n = 2a \quad \text{and} \quad m = 2b$$

where a and b are integers. Consider their subtraction $n - m$:

$$\begin{aligned} n - m &= 2a - 2b \\ &= 2\left(a - b\right) \qquad \left[\text{Factorizing}\right] \end{aligned}$$

We have $n - m$ is of the form 2(An Integer). By applying Definition (I.1) in the $\Leftarrow$ direction we conclude that $n - m$ is even.

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(c) *Proof.* We assume m and n are odd. By Definition (I.3) they can be written as

$$n = 2a + 1 \text{ and } m = 2b + 1$$

where a and b are integers. Consider $n - m$:

$$\begin{aligned} n - m &= (2a+1) - (2b+1) \\ &= 2a - 2b \\ &= 2(a-b) \qquad \text{[Factorizing]} \end{aligned}$$

We have $n - m$ is of the form 2(An Integer). By applying Definition (I.1) in the $\Leftarrow$ direction we conclude that $n - m$ is even whenever m and n are odd.

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(d) *Proof.* Let n be an odd number then by (I.3) there is an integer m such that $n = 2m + 1$. Consider n^2:

$$\begin{aligned} n^2 &= (2m+1)^2 \\ &= (2m+1)(2m+1) \\ &= 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1 \quad [\text{Rewriting } 4 = 2(2)] \end{aligned}$$

We have $n^2 = 2(\text{An Integer}) + 1$. By applying Definition (I.3) in the $\Leftarrow$ direction we conclude that n^2 is odd.

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(e) *Proof.* Let n be even. Then by Definition (I.1) this can be written as

$$n = 2a \quad \text{where } a \text{ is an integer}.$$

Let m be odd then by (I.3) this can be written as

$$m = 2b + 1 \quad \text{where } b \text{ is an integer}.$$

Consider $n + m$:

$$n + m = \underbrace{2a}_{=a} + \underbrace{2b+1}_{=m} = 2(a+b) + 1 \qquad [2(\text{An Integer}) + 1].$$

We have $n + m$ is 2(Integer) +1 therefore by (I.3) the number $n + m$ is odd.

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(f) *Proof.* Let n be an odd number then by (I.3) there is an integer a such that $n = 2a + 1$. Similarly let m be an odd number then there is an integer b such that $m = 2b + 1$. Consider their product nm:

$$\begin{aligned} nm &= (2a+1)(2b+1) \\ &= 4ab + 2a + 2b + 1 \\ &= 2(2ab + a + b) + 1 \quad [2(\text{An Integer}) + 1] \end{aligned}$$

We have $nm = 2(\text{Integer}) + 1$. By applying Definition (I.3) in the $\Leftarrow$ direction we conclude that the product nm is odd.

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(g) *Proof.* Since m is even we can write this as

$$m = 2k \text{ where } k \text{ is an integer.}$$

The product nm is given by

$$nm = n\left(2k\right) = 2kn\,.$$

Hence nm is a multiple of 2 therefore by Definition (I.1) we conclude that nm is even.

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3. (i) n is odd $\Rightarrow$ $n+1$ is even.

Proof. We assume n is odd. We know n and 1 are both odd therefore by Proposition (I.4) we have $n+1$ is even.

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(ii) For any integer n we have to show $n\left(n+1\right)$ is even because n and $n+1$ are consecutive integers.

Proof. If n is even then by the result of question 2(g) we have $n\left(n+1\right)$ is even. However, if n is odd then by the result of question 3(i) we have $n+1$ is even. Hence again by Question 2(g) we have $n\left(n+1\right)$ is even.

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4. If n is odd then n^3-1 is even.

Proof. By the result of question 2(d) we have n is odd implies n^2 is odd. Similarly by the result of Question 2(f) we have n^2 is odd implies nn^2 is odd. Hence $nn^2 = n^3$ is odd. Since n^3 and 1 are odd therefore by the result of question 2(c) we have n^3-1 is even. This completes our proof.

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5. (a) We need to prove $a \mid 0$.

Proof. We have $a \times 0 = 0$ therefore by Definition (I.5) we have $a \mid 0$.

(b) We need to prove $a \mid a$.

Proof. We have $a \times 1 = a$ therefore by Definition (I.5) we have $a \mid a$.

(c) We need to prove $1 \mid a$.

Proof. Since $1 \times a = a$ so by Definition (I.5) we have $1 \mid a$.

(d) Prove $a \mid a^2$.

Proof. Since $a \times a = a^2$ so by Definition (I.5) we have $a \mid a^2$.

(e) Prove $a \mid a^n$.

Proof. Since $a \times a^{n-1} = a^n$ which is $a \times \left(\text{Integer}\right) = a^n$ so by Definition (I.5) we have $a \mid a^n$.

(f) We have to prove $a \mid b$ and $a \mid c \Rightarrow a \mid \left(b+c\right)$.

Proof. We have $a \mid b$ and $a \mid c$ then

$$ax = b \text{ and } ay = c\,.$$

Therefore, adding these gives

$$b + c = ax + ay = a(x + y).$$

We have $a(x+y) = b + c$ which implies $a \mid (b+c)$ and this completes our proof.

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(g) Need to prove: $a \mid b$ and $a \mid c \quad \Rightarrow \quad a^2 \mid bc$

Proof. From $a \mid b$ and $a \mid c$ there are integers x and y such that

$$ax = b \text{ and } ay = c\,.$$

Multiplying these together gives

$$ax(ay) = bc \quad \text{which simplifies to} \quad a^2(xy) = bc\,.$$

Since $a^2(\text{Integer}) = bc$ therefore $a^2 \mid bc$.

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(h) Need to prove: $ac \mid bc \Rightarrow a \mid b$ where $c \neq 0$.

Proof. By using Definition (I.5) on $ac \mid bc$ we have there is an integer, x, such that

$$ac(x) = bc\,.$$

Dividing through by $c \neq 0$ gives

$$a(x) = b \text{ which implies } a \mid b\,.$$

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(i) Prove $a \mid b$ and $c \mid d \quad \Rightarrow \quad ac \mid bd$.

Proof. From $a \mid b$ and $c \mid d$ we have integers x and y such that

$$ax = b \text{ and } cy = d\,.$$

Multiplying these together gives

$$ax(cy) = bd$$
$$ac(xy) = bd$$
$$ac(xy) = cd \quad \text{which is } ac(\text{Integer}) = bd\,.$$

By using Definition (I.5) in the direction $\Leftarrow$ we have $ac \mid bd$ which is what was required.

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6. (a) We need to prove 'If n is odd then $8 \mid (n^2 - 1)$.'

Proof. We assume n is odd so it can be written as $n = 2m + 1$ where m is an integer. Consider $n^2 - 1$:

$$n^2 - 1 = (2m+1)^2 - 1$$
$$= \Big(\underbrace{4m^2 + 4m + 1}_{=(2m+1)^2}\Big) - 1 \qquad \text{[Expanding]}$$
$$= 4m^2 + 4m = 4m(m+1) \qquad \text{[Factorizing]}$$

We know by Question 3(ii) that $m(m+1)$ is even therefore we have

$$n^2 - 1 = 4\underbrace{m(m+1)}_{\text{Even}}$$

By Definition (I.1) we can write $m(m+1) = 2k$ where k is an integer. Hence, we have

$$n^2 - 1 = 4\underbrace{m(m+1)}_{=2k \text{ (Even)}}$$
$$= 4(2k) = 8k$$

Since $n^2 - 1 = 8k$ which means $8(\text{Integer}) = n^2 - 1$, therefore $8 \,\big|\, (n^2 - 1)$ and this completes our proof.

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(b) We need to prove 'If n is odd then $32 \,\big|\, (n^2+3)(n^2+7)$'.

Proof. We assume n is odd so it can be written as $n = 2m+1$ where m is an integer. Consider the first term $n^2 + 3$ and substituting $n = 2m+1$ into this yields:

$$n^2 + 3 = (2m+1)^2 + 3$$
$$= (4m^2 + 4m + 1) + 3 \qquad \left[\text{Expanding } (2m+1)^2\right]$$
$$= 4m^2 + 4m + 4 = 4(m^2 + m + 1) \qquad \text{[Factorizing]}$$

Similarly consider the second term $n^2 + 7$:

$$n^2 + 7 = (2m+1)^2 + 7$$
$$= (4m^2 + 4m + 1) + 7$$
$$= 4m^2 + 4m + 8 = 4(m^2 + m + 2)$$

Multiplying these together gives

$$(n^2+3)(n^2+7) = \underbrace{4(m^2+m+1)}_{=n^2+3}\underbrace{4(m^2+m+2)}_{=n^2+7}$$
$$= \underbrace{16}_{=4\times 4}(m^2+m+1)(m^2+m+2)$$

Let $m^2 + m + 1 = k$ where k is an integer. Substituting this into the above we have

$$(n^2+3)(n^2+7) = 16\underbrace{(m^2+m+1)}_{=k}\Big(\underbrace{m^2+m+1}_{=k}+1\Big)$$
$$= 16k(k+1)$$

By question 3(ii) we have $k(k+1)$ is even therefore we can write $k(k+1)=2\ell$ where ℓ is an integer. We have

$$\begin{aligned}(n^2+3)(n^2+7) &= 16(2\ell) \qquad \left[\text{Substituting } k(k+1)=2\ell\right]\\ &= 32\ell\end{aligned}$$

We have $32(\text{Integer})=(n^2+3)(n^2+7)$. By Definition (I.5) we conclude that

$$32 \,\Big|\, (n^2+3)(n^2+7).$$

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7. Show that if the last digit of an integer n is even then n is even.

Proof. Using the hint we have

$$\begin{aligned}n &= (a_m\times 10^m)+(a_{m-1}\times 10^{m-1})+(a_{m-2}\times 10^{m-2})+\ldots+(a_2\times 10^2)+(a_1\times 10^1)+a_0\\ &= \left[10(a_m\times 10^{m-1})+10(a_{m-1}\times 10^{m-2})+10(a_{m-2}\times 10^{m-3})+\ldots+10(a_2\times 10^1)+10(a_1)\right]+a_0\\ &\qquad\qquad \left[\text{Taking Out a Factor of } 10\right]\\ &= \left[\begin{array}{l}(2\times 5)(a_m\times 10^{m-1})+(2\times 5)(a_{m-1}\times 10^{m-2})+(2\times 5)(a_{m-2}\times 10^{m-3})+\ldots\\ \qquad +(2\times 5)(a_2\times 10^1)+(2\times 5)(a_1)\end{array}\right]+a_0\\ &\qquad\qquad \left[\text{Rewriting } 10 \text{ as } (2\times 5)\right]\\ &= 2\left[\begin{array}{l}5(a_m\times 10^{m-1})+5(a_{m-1}\times 10^{m-2})+5(a_{m-2}\times 10^{m-3})+\ldots\\ \qquad +5(a_2\times 10^1)+5(a_1)\end{array}\right]+a_0\end{aligned}$$

The last line says $n=2\left[\text{An Integer}\right]+a_0$. We assume a_0 is even because the given proposition says "if the last digit of an integer n is even" and a_0 is the last digit. We can write $a_0=2b$. We have

$$\begin{aligned}n &= 2\left[\text{An Integer}\right]+a_0\\ &= 2\left[\text{An Integer}\right]+2b = 2\left(\left[\text{An Integer}\right]+1\right)\end{aligned}$$

$(\text{An Integer}+1)=(\text{Another Integer})$ therefore

$$n = 2(\text{Another Integer})$$

and so by (I.1) we conclude that n is even.

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8. Show that if the last digit of an integer n is odd then n is odd.

Proof. Very similar to the proof of Question 7.