Exercises I.2

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Throughout this exercise lower case letters such as a, b, c,…, n,… *represent integers.*

1. Show that the following are tautologies.

(a) $(\neg P) \vee P$

(b) $[(P \Rightarrow Q) \wedge (P \Rightarrow R)] \Rightarrow [P \Rightarrow (Q \wedge R)]$

(c) $[(P \Rightarrow Q) \wedge (R \Rightarrow Q)] \Rightarrow [(P \vee R) \Rightarrow Q]$

(d) $[(P \Rightarrow Q) \wedge (\neg Q)] \Rightarrow (\neg P)$

2. Prove the following propositions:

(a) If m and n are even then their sum $n+m$ is even.

(b) If m and n are even then their subtraction $n-m$ is even.

(c) If m and n are odd then their subtraction $n-m$ is even.

(d) If n is an odd number then n^2 is also odd.

(e) If m is even and n is odd then their sum $m+n$ is odd.

(f) If m is odd and n is odd then their product $n \times m = nm$ is odd.

(g) If n is any integer and m is even then their product $n \times m = nm$ is even.

3. Prove the following propositions:

(i) n is odd $\Rightarrow$ $n+1$ is even

(ii) *The product of two consecutive integers is even.

[Hint: Use Proposition (I.4)].

4. Prove that if n is odd then $n^3 - 1$ is even.

[Hint: Use the propositions proved in Question 2].

5. Prove the following propositions:

(a) $a \mid 0$ (b) $a \mid a$ (c) $1 \mid a$ (d) $a \mid a^2$

(e) $a \mid a^n$ where $n \geq 1$ is an integer.

(f) $a \mid b$ and $a \mid c$ $\Rightarrow$ $a \mid (b+c)$

(g) $a \mid b$ and $a \mid c$ $\Rightarrow$ $a^2 \mid bc$

(h) $ac \mid bc$ $\Rightarrow$ $a \mid b$ where $c \neq 0$.

(i) $a \mid b$ and $c \mid d \Rightarrow ac \mid bd$

The remaining three questions are more difficult.

6. Prove the following proposition:

If n is odd then

(a) $8 \mid (n^2 - 1)$ (b) $32 \mid (n^2 + 3)(n^2 + 7)$

7. Show that if the last digit of an integer n is even then n is even.

[Hint: Write n as

$$n = a_m a_{m-1} a_{m-2} ... a_2 a_1 a_0 \quad (m > 1)$$

where a_m, a_{m-1}, $a_{m-2}, ...,$ a_2, a_1 and a_0 are the digits of n. Note that a_0 is the last digit. Hence n can be expressed as:

$$n = (a_m \times 10^m) + (a_{m-1} \times 10^{m-1}) + (a_{m-2} \times 10^{m-2}) + ... + (a_2 \times 10^2) + (a_1 \times 10^1) + a_0]$$

8. Show that if the last digit of an integer n is odd then n is odd.