

Exercises I.4

Complete solutions at www.oup.co.uk/companion/NumberTheory

1. Show that for all-natural numbers, n :

$$2 + 4 + 6 + \dots + 2n = \sum_{m=1}^n 2m = n(n+1).$$

2. Prove that for all-natural numbers, n :

$$2 + 5 + 8 + \dots + (3n - 1) = \sum_{m=1}^n (3m - 1) = \frac{1}{2}n(3n + 1).$$

3. Prove that for all-natural numbers, n :

$$\sum_{m=1}^n m^3 = \frac{1}{4}n^2(n+1)^2.$$

4. *Prove that for all natural numbers, n :

$$\sum_{m=1}^n m^3 = (1 + 2 + 3 + 4 + \dots + n)^2.$$

[Hint: Use the result of Question 3].

5. Prove that for all natural numbers, n :

$$(1 \times 2) + (2 \times 3) + \dots + n(n+1) = \sum_{m=1}^n m(m+1) = \frac{1}{3}n(n+1)(n+2).$$

6. Prove that for all natural numbers, n :

$$(1 \times 2 \times 3) + (2 \times 3 \times 4) + \dots + n(n+1)(n+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$$

7. Show that for all natural numbers, n :

$$\sum_{m=0}^n 2^m = 2^{n+1} - 1.$$

8. *Prove that for all natural numbers, n :

$$\sum_{m=1}^n (2m-1)^3 = n^2(2n^2-1).$$

9. Prove that for all natural numbers, n :

$$\sum_{m=1}^n m^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

10. Prove that for all natural numbers, n :

$$\sum_{m=1}^n m^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}.$$

11. Prove that for all natural numbers n , 9 divides $10^n - 1$.

12. Prove that for all natural numbers n :

$$3 \mid (n^3 - n).$$

13. *Show that for every natural number n :

$$3 \mid n(n+1)(n+2).$$

[This means that 3 divides three consecutive integers.]

14. Prove that for all natural numbers n :

$$n^2 - n \text{ is an even number.}$$

15. (i) Prove that for all natural numbers n :

$$\sum_{m=1}^n ar^{m-1} = a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(1-r^n)}{1-r} \quad [r \neq 1]$$

where a and r are real numbers.

[This is a geometric series with first term equal to a and common ratio r .]

(ii) Show that for every natural number, n :

$$1 + r + r^2 + \dots + r^n = \sum_{m=0}^n r^m = \frac{1-r^{n+1}}{1-r} \quad (r \neq 1)$$

[This is the geometric series with the first term equal to 1.]

16. **Prove that for all natural numbers n we have the following trigonometric identity:

$$\sin(x) + \sin(2x) + \dots + \sin(nx) = \frac{\cos\left(\frac{x}{2}\right) - \cos\left(\frac{2n+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)}.$$

where x is a real number such that $\sin\left(\frac{x}{2}\right) \neq 0$. [You will need to know your trigonometric identities to do this question or at least look them up and apply them.]

17. ***Prove the binomial theorem for the natural number n :

If a and b are real numbers, then the binomial theorem says that for all natural numbers n :

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \dots + b^n.$$