Exercises **I.5**

Brief solutions end of Exercises.

Complete solutions at www.oup.co.uk/companion/NumberTheory

1. Determine the members of the following sets:

(a) $A = \left\{x \in \mathbb{R} : 2x + 1 = 0\right\}$ (b) $B = \left\{x \in \mathbb{N} : (x-1)(x-2) = 0\right\}$

(c) $C = \left\{x \in \mathbb{N} : x^2 - 2x + 1 = 0\right\}$ (d) $D = \left\{x \in \mathbb{Z} : x^2 - 4x - 5 = 0\right\}$

(e) $E = \left\{x \in \mathbb{Z} : x^2 - 9 = 0\right\}$ (f) $F = \left\{x : x \text{ is a prime number less than } 10\right\}$

2. Determine the elements of the following sets:

(a) $A = \left\{x \in \mathbb{N} : (x-1)(x+3) = 0\right\}$ (b) $B = \left\{x \in \mathbb{N} : 2x + 1 = 0\right\}$

(c) $C = \left\{x \in \mathbb{Z} : (x+5)(3x-1) = 0\right\}$ (d) $D = \left\{x \in \mathbb{Q} : (x+5)(3x-1) = 0\right\}$

(e) $E = \left\{x \in \mathbb{Q} : x - \pi = 0\right\}$ (f) $F = \left\{x \in \mathbb{R} : x - \pi = 0\right\}$

3. Write the following statements in set notation:

(a) The set of negative real numbers.

(b) The set of positive integers.

(c) The set of real numbers between 0 and 2 excluding 0 and 2.

(d) The set of rational numbers less than 1.

(e) The set of natural numbers which are multiples of 10.

4. On a Venn diagram shade in the following regions.

(a) A^c (b) B^c (c) $\left(A \cap B\right)^c$ (d) $\left(A \cup B\right)^c$ (e) $A^c \cap B^c$

What do you notice about your results to parts (d) and (e).

5. By shading in the appropriate area on a Venn diagram show that $\left(A^c\right)^c = A$.

6. Determine in each of the following cases whether $A \subseteq B$ or $B \subseteq A$:

(a) $A = \left\{x \in \mathbb{R} : 0 \le x \le 10\right\}$ and $B = \left\{x \in \mathbb{R} : -10 \le x \le 10\right\}$.

(b) $A = \left\{x \in \mathbb{Z} : 0 \le x \le 10\right\}$ and $B = \left\{x \in \mathbb{R} : 0 \le x \le 10\right\}$.

(c) $A = \left\{x \in \mathbb{R} : 0 \le x \le 10\right\}$ and $B = \left\{x \in \mathbb{Z} : 0 \le x \le 10\right\}$.

(d) $A=\{x\in\mathbb{N}:-10\le x\le 10\}$ and $B=\{x\in\mathbb{Z}:1\le x\le 10\}$.

(e) $A=\{x\in\mathbb{Q}:0\le x\le 10\}$ and $B=\{x\in\mathbb{Q}:0<x<10\}$.

7. Let

$$\varnothing,\ A=\{2,\ 3,\ 5\},\ B=\{x\in\mathbb{Z}:x^2-4=0\}$$
$$C=\{x\in\mathbb{N}:x \text{ is prime and less than } 10\},\ D=\{x\in\mathbb{Z}:0\le x\le 10\}$$

Decide whether the following pairs of sets are subsets or not:

(a) $\varnothing,\ A$ (b) $A,\ A$ (c) $A,\ C$ (d) $C,\ D$ (e) $B,\ C$

8. Consider the following sets:

$$A=\{x\in\mathbb{Z}:0<x<5\},\ B=\{x\in\mathbb{N}:x \text{ is an even number}\}$$
$$C=\{x\in\mathbb{N}:x \text{ is a multiple of } 2\}$$
$$D=\{x\in\mathbb{R}:x\ne x\},\ E=\{x\in\mathbb{N}:x^3\}$$
$$F=\{x\in\mathbb{Z}:0<x<2\}$$

Determine whether the symbol □ in the following is $\subseteq$ or $\not\subseteq$:

(a) $A \,\square\, B$ (b) $A \,\square\, C$ (c) $B \,\square\, C$ (d) $C \,\square\, B$ (e) $A \,\square\, D$

(f) $D \,\square\, A$ (g) $E \,\square\, F$ (h) $F \,\square\, E$

9. Determine the cardinality, $|A|$, of the following sets:

(a) $A=\varnothing$ (b) $A=\{a,\ b,\ c\}$ (c) $A=\{x\in\mathbb{Z}:3x^2-x=0\}$

(d) $A=\{x\in\mathbb{R}:x=x+1\}$

10. Let $A=\{1,\ 2,\ 3,\ 4,\ 5\}$ and $B=\{x\in\mathbb{N}:x \text{ is a prime number } \le 5\}$. Show that $A\not\subseteq B$ but $B\subseteq A$.

11. Let $A=\{1,\ 3\}$, $B=\{1,\ 3,\ 3,\ 1\}$ and $C=\left\{1,\ 3,\ \frac{3}{1},\ \frac{\pi}{\pi}\right\}$. Determine a set relationship between the given sets $A,\ B$ and C.

12. Use the Well Ordering Principle to prove that for all positive integers n:

$$\sum_{m=1}^{n} m=\frac{n(n+1)}{2}$$

[We proved this result by induction in the last section, Example 29.]

13. *Prove the Induction Principle (I.15) by using the well ordering principle.

This results proves that the Induction Principle and WOP are equivalent because we have already derived the WOP from the Induction Principle.

Brief Solutions to Exercise I.5

1. (a) $A = \left\{-\frac{1}{2}\right\}$ (b) $B = \{1,\ 2\}$ (c) $C = \{1\}$ (d) $D = \{-1,\ 5\}$

(e) $E = \{-3,\ 3\}$ (f) $F = \{2,\ 3,\ 5,\ 7\}$

2. (a) $A = \{1\}$ (b) $B = \varnothing$ (c) $C = \{-5\}$ (d) $D = \left\{\frac{1}{3},\ -5\right\}$

(e) $E = \varnothing$ (f) $F = \{\pi\}$

3. (a) $\{x \in \mathbb{R} : x < 0\}$ (b) $\{x \in \mathbb{Z} : x > 0\}$ or $\mathbb{N}$ (c) $\{x \in \mathbb{R} : 0 < x < 2\}$

(d) $\{x \in \mathbb{Q} : x < 1\}$ (e) $\{10n : n \in \mathbb{N}\}$

6. (a) $A \subseteq B$ (b) $A \subseteq B$ (c) $B \subseteq A$ (d) $A \subseteq B$ and $B \subseteq A$ which means we have $A = B$ (e) $B \subseteq A$

7. (a) $\varnothing \subseteq A$ (b) $A \subseteq A$ (c) $A \subseteq C$ (d) $C \subseteq D$ (e) $B \not\subseteq C$

8. (a) $A \not\subseteq B$ (b) $A \not\subseteq C$ (c) $B \subseteq C$ (d) $C \subseteq B$ (e) $A \not\subseteq D$

(f) $D \subseteq A$ (g) $E \not\subseteq F$ (h) $F \subseteq E$

9. (a) $|\varnothing| = 0$ (b) $|A| = 3$ (c) $|A| = 1$ (d) $|A| = 0$

11. $A = B = C$