Exercises **I.3**

Brief solutions end of Exercises.

Complete solutions at www.oup.co.uk/companion/NumberTheory

1. By constructing the truth table show that

$$\left[\text{not } \left(P \Rightarrow Q\right)\right] \equiv \left[P \wedge \left(\text{not } \ Q\right)\right] \qquad \qquad [\text{Equivalent}]$$

2. Prove the following results:

(a) $x^2 - 3x + 2 = 0 \quad \Leftrightarrow \quad x = 1 \text{ or } x = 2$

(b) $x^2 - 10x + 21 = 0 \quad \Leftrightarrow \quad x = 3 \text{ or } x = 7$

(c) $x^2 - 1 = 0 \quad \Leftrightarrow \quad x = 1 \text{ or } x = -1$

(d) $x^2 - \left(a + b\right)x + ab = 0 \quad \Leftrightarrow \quad x = a \text{ or } x = b$

(e) $x^2 = y^2 \quad \Leftrightarrow \quad x = y \text{ or } x = -y$

3. Prove the following propositions:

(a) n is even $\Leftrightarrow$ n^2 is even.

(b) mn is odd $\Leftrightarrow$ both m and n are odd.

(c) $m + n$ is odd $\Leftrightarrow$ only m or only n is odd.

(d) mn is even $\Leftrightarrow$ at least one of m or n is even.

4. Let P and Q represent the following mathematical propositions. In each case decide whether $P \Rightarrow Q$ or $Q \Rightarrow P$ or $P \Leftrightarrow Q$. You do *not* have to prove any of these statements:

(a) $P:\ a^2 > 0, \quad Q:\ a > 0$

(b) $P:\ a \neq 0, \quad Q:\ a^2 > 0$

(c) $P:\ x > 3, \quad Q:\ x > 4$

(d) $P: x = 2 \text{ or } x = -1, \quad Q: x^2 - x - 2 = 0$

(e) $P: ax^2 + bx + c = 0$ has two real roots, $\quad Q:\ b^2 - 4ac \geq 0$

(f) $P:\ a \mid \left(b + c\right), \quad Q:\ a \mid b \text{ and } a \mid c$

(g) $P:\ ac \mid bc, \quad Q:\ a \mid b$ where $c \neq 0$.

(h) $P:\ e^x = 1, \quad Q:\ x = 0$

(i) $P:\ \ln\left(x\right)=0, \quad Q:\ x=1$

(j) $P:\ 0<a<b, \quad Q:\ a^n<b^n$ where n is a positive integer.

(k) $P:\ 0<x<y, \quad Q:\ 0<\frac{1}{y}<\frac{1}{x}$

5. Consider the following four cards:

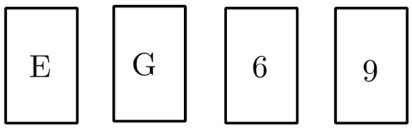

Figure 5

Each card has a letter on one side and a number on the other. You are told that if a card has a vowel on one side then it has an even number on the other. Which two cards should you turn over to test this statement?

6. In a class there are eight students. Prove that at least two of the students were born on the same day of the week.

In each case prove the following statements by applying contradiction. In some cases it may be easier to do a direct proof but this is an exercise in proof by contradiction.

7. Prove the following proposition:

For every real number, x, there is a unique real number y such that

$$x+y=0$$

[y is called the **additive inverse** of x].

8. Let x and y be real numbers. Prove that

$$xy=0 \quad \Rightarrow \quad x=0 \quad \text{or} \quad y=0$$

In the remaining questions lower case letters represents an integer.

9. Prove that n^2 is odd $\Rightarrow$ n is odd.

[We have proved this result by contrapositive in Example 22. This time prove the result by contradiction and compare the two proofs].

10. Prove that n^3 is odd $\Rightarrow$ n is odd.

11. Prove that n^3 is even $\Rightarrow$ n is even.

12. Prove that ab is odd $\Rightarrow$ both a is odd and b is odd.

13. Prove that ab is even $\Rightarrow$ a is even or b is even.

14. Prove that $\sqrt{6}$ is irrational.

15. Prove that $\sqrt[3]{2}$ is irrational.

16. Prove that there are *no* positive integer solutions such that

$$a^2 - b^2 = 1.$$

17. (i) Prove that the sum of a rational and irrational number is irrational.

(ii) Prove that $a + b\sqrt{n}$ is irrational if n is a non- square number and a and $b \neq 0$ are integers.

18. Consider the triangle shown in Fig 6. Show that if $\angle B = \angle C$ then $AB = AC$.

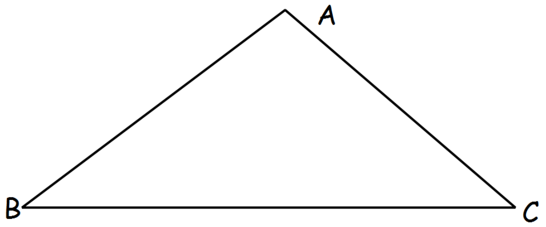

Figure 6

Brief Solutions to Exercises I.3

4. (a) $Q \Rightarrow P$ (b) $P \Leftrightarrow Q$ (c) $Q \Rightarrow P$

(d) $P \Leftrightarrow Q$ (e) $P \Leftrightarrow Q$ (f) $Q \Rightarrow P$

(g) $P \Leftrightarrow Q$ (h) $P \Leftrightarrow Q$ (i) $P \Leftrightarrow Q$

(j) $P \Leftrightarrow Q$ (k) $P \Leftrightarrow Q$

5. Cards E and 9.