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A Brief Survey of Euler's Mathematics: The Basel Problem

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1 Acknowledgements

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2 Lay Summary

The Basel Problem is a specific infinite series that bewildered a number of mathematicians before Euler. An infinite series is an infinite number of terms that are added together to find the exact value that the sequence approaches. This exact value can be a finite value or an infinite value. An infinite series can be defined using the sigma notation, with an expression that can be expanded using the indicated values of n.

$$\sum_{n=1}^{\infty} a_n$$

The Basel Problem can be written using the same sigma notation:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = 1 + \frac{1}{4} + \frac{1}{9} + \ldots$$

An infinite series is said to converge if the total of the terms is a finite value, therefore the Basel Problem is a convergent series as the exact value is $\pi^2/6$. If the value is infinite, the series is said to be divergent. So Euler decided to pursue the Basel Problem as it was known that the series had an exact value from the work of the mathematicians prior to him. As the series is known to converge extremely slowly, adding as many terms as possible will not provide an exact answer. Therefore, Euler had to devise a proof using a range of mathematical properties.

The Basel Problem includes many mathematical elements, including Euler's number, trigonometric functions, logarithms and π. Euler's number, e, is an irrational number that can be represented in the form of an infinite series. The number is applied in real world terms, such as interest rates in banking, rates of decay in science and population models for systems. The function e^x is significant as it is the inverse function, $ln(x)$, is present in Euler's attempts at solving the Basel Problem for it's exact value.

Furthermore, the function $ln(x)$ can also be described as an infinite series. However, this series converges slowly so it is almost useless to sum the values as it takes a large amount of terms to find an additional decimal place for the approximation. The logarithm is defined as $log_b(x)$, meaning the logarithm finds the power that the base, b, needs to be raised to in order to obtain the value x. The natural logarithm, $ln(x) = log_e(x)$, means the base b is Euler's number itself, e. The natural logarithm is not defined at $x = 0$ so the Taylor series expansion is taken at the point $x = 1$.

$$ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$$

π can be represented as the infinite series of four times the expansion of $tan^{-1}(1)$. π is present in the Basel Problem, and it was Euler that connected the exact value to π. The Basel Problem's summation is $\pi^2/6$ which implies Euler's recognition of the value was the cause of his findings, as other mathematicians failed to identify the presence of π.

From the revelation of the Basel Problem, the Riemann zeta function was produced. The Riemann zeta function is the function $\zeta(s)$ where the sum of the series is calculated for when s is the power of the reciprocal of the natural numbers $n = 1, 2, 3, \ldots$ denoted as $\sum_{n=1}^{\infty} \zeta(s) = 1 + 1/2^s + 1/3^s + \ldots$.

The report will explain all elements described and use them to produce proofs of the Basel Problem. After identifying the Basel Problem, the effects of Euler's solution will be observed to show the impact Euler's discoveries had on mathematics.

3 Introduction

Euler was a remarkably talented Swiss mathematician who lived throughout the 1700's and his work is still currently in use. He had a strong career and his talent as a mathematician was noticed early on in his life. He produced discoveries in several areas of mathematics such as number theory, calculus, complex analysis, geometry, mechanics and physics. By exploring Euler's younger years and time at the academies in St. Petersburg and Berlin, the impact of the political state can be acknowledged as it hugely effected his career and work in mathematics. During this era, Euler was a considerably popular mathematician and was provided a platform through both the St. Petersburg Academy and the Berlin Academy to publish his life work. He published a multitude of pieces, developed many theories and even began the foundations of many mathematical problems.

This investigation will look at the background of the mathematician himself and observe the theories and people that influenced him and the impact he had on future discoveries. This investigation will look into the use of logarithms and Euler's notation of the function, which had a massive significance on future mathematics. It will also inspect his work on infinite series by the use of functions and how he popularised the use of π. By combining these sections, the famous Basel Problem is formed.

The Basel Problem is an infinite series summation of the reciprocal of the square of the natural numbers. This investigation will explore the discoveries before Euler's attempt, as plenty of mathematicians had attacked the problem to find the exact solution but failed, and only found a rough boundary or rounded decimal. Within this investigation, Jacob Bernoulli's attempt to find an exact value resulted in him finding an upper bound. This proof will be explained, presenting a starting point for Euler as it enabled him to find an estimate as he knew from Bernoulli's proof that the exact value is less than 2. This project will uncover how Euler was motivated and influenced to find the exact value $\pi^2/6$ from the findings prior.

When detailing Euler's number of proofs for the Basel Problem, it will begin with how he estimated the value, and end with a derivation of a proof to find the exact value $\pi^2/6$. The investigation will detail the computations on how the value was found and consequently, how Euler extended his research. Additionally, it will look at how the Riemann zeta function was formed as an advancement of the Basel Problem. Not only was the Riemann Zeta function a following result from the Basel Problem, another following result was Euler's encounter with series of the same form. Euler was able to derive a general formula for $\sum_{n=1}^{\infty} 1/n^s$ when s is even. However, Euler was unable to determine a general formula for the case when s is odd. He also could not determine an exact value for $\sum_{n=1}^{\infty} 1/n^3$, although, the investigation will review mathematicians that develop Euler's work to establish the answer.

4 History of Leonhard Euler

4.1 Early Life & Beginning of Education

> “The 18th century was sometimes referred to as the ‘Era of Euler’.”

(McElroy 2014)

Leonhard Euler – arguably the greatest mathematician of all time – was born on April 15th, 1707 near Basel, Switzerland (Dunham 2007). Euler came from a highly religious family with his maternal grandfather working as a pastor, and his father a pastor of the reformed church in a village near Basel called Riehen (Dunham 2007). With religious paths on each side of the family, Euler’s father anticipated that Leonhard would pursue a career similar to himself. However, this was not the case. Leonhard Euler had a natural talent for mathematics from a young age and an incredible memory.

His father, Paul III Euler, studied mathematics in addition to other subjects at the University of Basel and gained a masters of art degree after altering his studies to theology (Calinger 2015). During Leonhard’s younger years, Paul III taught him mathematics from a young age until he attended the Basel Gymnasium, that taught Latin, Greek and ancient classics (Bradley and E. Sandifer 2007). The Euler’s then employed a novice mathematician to tutor the young Leonhard in mathematics and humanities (McElroy 2014; Bradley and E. Sandifer 2007). He memorised prime powers and acquired the flair for languages, evidently because his mother had familiarised him with Greek and Roman (Bradley and E. Sandifer 2007).

Euler’s mental mathematics was exceptional. His raw talent led to him enrolling at the University of Basel in 1720 where he connected with Johann Bernoulli (Dunham 1999). Euler studied other subjects rather than purely mathematics such as Oriental languages and theology (Dunham 2007). Bernoulli was a highly talented mathematician who came from a family destined to be mathematicians (Dunham 1999). Leonhard’s father Paul, was a family friend of the Bernoulli’s and had attended lectures given by Jacob Bernoulli, the brother of Johann (McElroy 2014). Additionally, throughout his time at university, Paul took up residence with Jacob and Johann Bernoulli and produced his ratio and proportions thesis under Jacob’s supervision (Bradley and E. Sandifer 2007). Johann Bernoulli became an advisor to Euler by providing a support system for Euler’s mathematical questions. Euler visited Bernoulli frequently and they met every Saturday afternoon to explore Euler’s queries. Bernoulli recommended that Euler peruse complex books on a variety of subjects such as mathematics, physics and astronomy (Bradley and E. Sandifer 2007). As the years went on, Euler’s intelligence developed and Bernoulli began to learn from him. Nicolaus and Daniel Bernoulli, Johann’s sons, became close friends of Leonhard Euler whilst at university.

In 1722, he achieved a bachelor of arts degree and in 1723 he then earned a master’s degree in philosophy with the subject of his dissertation being a comparison between Newton and Descartes philosophies. Alternatively, with his father’s influence, Euler began to study theology to lead the way into the ministry. When he realised that he wanted to pursue mathematics further, Euler left the ministry (Dunham 1999). Euler began his research in the field of mathematics and even released a document on algebraic reciprocal trajectories (McElroy 2014). Euler composed multiple dissertations, one on the “nature and propagation of sound" and another about ship masts that he was awarded second place from the French Academy of Sciences (Dunham 2007). With Euler residing in Switzerland, the discovery of the masts of ships was extraordinary considering he was located nowhere near the ocean. So instead he had to examine boats on the Rhine rather than ships on the ocean (Bradley and E. Sandifer 2007). He was a profoundly educated man who wrote about multiple subjects during his time in education, like the history of law and physiology (Dunham 1999; Dunham 2007).

The University of Basel fell into a financial crisis in the course of Euler's attendance. The number of students enrolling at the university dropped significantly from a thousand to roughly a hundred with only nineteen lecturers (Bradley and E. Sandifer 2007). The lecturers salary was abysmal and the quality of the professors was of a low standard with the exception of Johann Bernoulli (Bradley and E. Sandifer 2007). Even so, Euler applied for the position of the Chair of Mathematics at the university, but his application was denied (Wilson 2007). This setback was beneficial for Euler, the university's quality had declined so he proceeded onto a more appropriate path within his career.

4.2 St. Petersburg Academy

In 1725, Nicolaus and Daniel Bernoulli accepted job invitations for The St. Petersburg Academy in Russia to teach (Suzuki 2009). Daniel Bernoulli became a professor in mathematics and Nicolaus a professor in physics. A year later, Euler accepted a job proposition also at St. Petersburg Academy to teach physiology and medicine on the recommendation and approval of the Bernoulli brothers who suggested Euler to Catherine I (Dunham 1999; Dunham 2007). Job opportunities were a minority in Switzerland and with the University of Basel on a financial decline, Euler accepted this offer with a blank mind as he knew nothing about the subject but required the job. As there was a job shortage, this likely pushed him to leave and he began educating himself medicine unaware that when he would reach the academy in 1727, the position would adjust to physics. Consequently, this was because Nicolaus had died shortly after arriving in Russia as he had acquired hectic fever (Suzuki 2009).

After Catherine I died, Peter II ruled Russia for 3 years until his death in 1730 at 14. The Russian government consisted of Russian nobles who considered the Academy to be filled with foreign lecturers and superiors. This aggravated the Russian government and the finances for the academy were cut off (Bradley and E. Sandifer 2007). This resulted in Euler taking a career break in academia and he became a medic for the Russian navy momentarily (Bradley and E. Sandifer 2007). Afterwards, Anna of Russia ruled for 10 years and the Academy's finance situation improved. Euler returned to the Academy to continue educating and refused a promotion from the navy (Bradley and E. Sandifer 2007). Euler and Daniel Bernoulli took up residence together whilst teaching at St. Petersburg Academy, allowing them to discuss their theories and subjects. In 1733, Daniel Bernoulli was offered a position back in Basel, Switzerland and decided to depart Russia (Dunham 1999). This meant the standing of professor of mathematics was left open, which Euler decided to occupy due to his passion for the topic.

A year after Daniel's departure, Euler married Katharina Gsell, whose father was a Swiss painter residing in Russia (Dunham 1999). Katharina conceived 13 children with Leonhard. However, due to the time period, children struggled to fight for survival and only 5 of their children lived to adulthood. Katharina cared for their two daughters and three sons whilst Leonhard focused on his career (Bradley and E. Sandifer 2007). Sadly, only 3 of their children outlived their parents, Leonhard and Katharina (Dunham 1999).

During Tsarina Anna's rule, Russia's political state declined rapidly. The academy was still vulnerable as the academy had a team of staff whose nationality was mostly non-Russian (Dunham 1999). Tsarina Anna's adviser and lover was Ernst Johann Biron, the Duke of Courland. Biron was not well liked and caused uproar due to his decision to killing and exiling thousands of people. Biron was eventually exchanged for Burkhard Münnich, a German who was initially in control of the Russian army (Suzuki 2009). Russian nobles were immensely displeased with the number of non-Russians in significant powerful positions (Suzuki 2009). Socially and politically, Euler was uncomfortable with his employment with the academy thus, Frederick the Great's promising proposal occurred at an ideal time.

Whilst teaching at St. Petersburg Academy, Euler had to serve the government but continued to research and produce theories in his spare time. The government asked him to be a scientific consultant and during his duties he "prepared maps, advised the Russian navy, and even tested designs for fire engines" (Dunham

1999). Therefore, Euler was greatly involved with the government so as the government changed their outlook on the Academy, Euler's position became strained. Euler continued to find his way through mathematics and communicated with other mathematicians around the world via letters.

Euler later discovered and published in 1740 the true value of the summation of the infinite series, which is now established as "The Basel Problem" which he found to be $\pi^2/6$. The Basel Problem is the summation of the inverse square integers.

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{k^2}+\cdots=1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{k^2}+\cdots=\frac{\pi^2}{6}$$

A number of mathematicians had attempted to find the exact value previous to Euler, with the correct value taking 84 years to finally appear as the problem was first raised in 1650 by Pietro Mengoli (Dunham 1999). Euler made multiple discoveries throughout his time in St. Petersburg such as his development of integral calculus and computation of the function sine (Dunham 2007).

4.3 Berlin Academy

Frederick the Great of Prussia, provided Euler with an offer he could not refuse. Due to the declination of political climate within Russia, Euler accepted a position at the Berlin Academy in 1741. Euler stayed in Berlin for 25 years and acquired the position of director of the mathematics sector (Bradley and E. Sandifer 2007). Politics in Russia became alarming due to the new leadership where Peter the Great's daughter Elizabeth interchanged Anna Leopoldovna, who was then exiled. Euler stated that he had "Just come from a country where a man's words can get him hanged" (Calinger 2015).

In 1742, Euler was granted Frederick's approval to tutor mathematics to the Duchess of Württemburg's sons as he was short of money due to the Prussian Society providing funds to the First Silesian War (Calinger 2015). He also began to tutor pupils of the Petersburg Academy of Sciences (Calinger 2015). During his time in Berlin, Euler was put in contact with the Princess Anhalt Dessau and educated her in the subject of science (Dunham 1999). The letters he penned were compiled and published with the name 'Letters of Euler on Different Subjects in Natural Philosophy Addressed to a German Princess' (Dunham 1999). The letters included a range of topics such as 'light, sound, gravity, logic, language, magnetism and astronomy' (Dunham 1999). The letters included Euler's obtained theories about the colour of the sky, and the wonder of vision, which is ironic written by a man with virtually no sight (Dunham 1999). He also discussed the moon's motion, the idea of tides, how a river flows and the vibration of strings and their motion (Wilson 2007).

Whilst in Berlin, Euler maintained his relationship with the St. Petersburg Academy and its staff so his entries in the St. Petersburg journal were continually added (Dunham 1999). In 1742, the Academy also provided Euler with a pension as the government maintained a strong relationship with him (Dunham 2007). Although, the Berlin Academy were also publishing other documents of his as he was producing such a large quantity of mathematical and scientific content (Bradley and E. Sandifer 2007). He also mailed files to the Paris Academy which resulted in him participating in the Paris Prize competition again (Bradley and E. Sandifer 2007). His relationship with Russia benefited him in the long run as in 1756, the Seven Years' War began as Russia attacked Prussia (Dunham 1999). A farm that belonged to Euler was destroyed because of the war outbreak, however when the Russian general found out, he ordered reparations to be made towards Euler (Dunham 2007). Furthermore, Elizabeth the Empress of Prussia compensated Euler for the damage done by the Russian troops (Dunham 2007). The correspondence between the boundaries of the countries lessened, but despite this, Euler sustained his mathematical network by continually sending his papers (Bradley and E. Sandifer 2007).

In 1748, he published "Introductio in Analysin Infinitorum" where he explained his idea of the number

$e = 2.71828...$ which is the summation of the reciprocal of the factorials of whole numbers.

$$\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

This is when Euler conveyed his finding on infinite series. He found his own identity $e^{ix} = cos(x) + isin(x)$ (Wilson 2007).

Slowly, Euler's good relationship with Frederick began to fade as Frederick became invested in the Academy and Euler found him "pretentious" (Wilson 2007). After the death of the president of the Royal Prussian Academy of Sciences, King Frederick controlled the Academy. Sadly, Euler was never considered for the position, even after all the work he did as director of mathematics (Bradley and E. Sandifer 2007). Euler previously worked alongside the president Pierre Louis Maupertuis, so would continue to work with fresh director, Frederick II, considering he had gained control (Bradley and E. Sandifer 2007). Although Euler carried out duties that the President should, he was never granted the official title. The currency value dropped in Berlin, effecting Euler's livelihood after the war. Therefore, the decision was made to return to St. Petersburg. It was difficult to obtain the King's permission for Euler to depart, as even though the King may not have held Euler in high regards as a person, he was not oblivious to Euler's raw talent. As a result of this, Euler reverted to abandoning all of his duties until finally, the King provided his approval and at the age of 59 he continued the journey he once began in St. Petersburg (Bradley and E. Sandifer 2007).

4.4 Eyesight Deterioration

The Euler's originally had the surname Euler-Schölpi. "Schölpi, which derives from the words schelb and schief, related to schielen, signifies squint-eyed, cross-eyed, or crooked" (Calinger 2015). The name implies that the Eulers were likely to gain eye conditions, however, from 1735 onwards, he experienced hectic fevers which presumably induced the issues he had with his eyes.

Euler's eyesight slowly deteriorated throughout his adult life. He became partially blind in his right eye in 1738 due to an abscess induced by a harsh infection (Calinger 2015). Despite Euler believing the damage was done from his research, writing and intricate work producing maps, the time period likely aided the loss of his eyesight due to easy contamination and poor sanitation. His reading and investigations made his eyes weak assisting to his sight loss.

Throughout his time in Berlin, Frederick the King of Prussia labelled Euler as "cyclops" – a remark referring to his poor eyesight (Dunham 1999). Additionally, his vision became blurred and he developed a cataract in his left eye in 1766 which was later operated on in an attempt to salvage his sight. The operation was short with no complications. Euler required bandages and was advised not to overwork or stress his eyes (Calinger 2015). One can only deem this impossible for the talented Euler. Ultimately, the operation was successful, even so, Euler contracted an infection directly from the large incision made in the course of the operation (Bradley and E. Sandifer 2007). His overall health took a decline with his hearing beginning to worsen (Calinger 2015).

In 1771, Euler became practically blind and could not "write or read anything other than very large characters" (Dunham 1999). Although he essentially lost his vision, his condition did not deter him from his exploration. During the year of 1771, he composed one mathematical paper per week by instructing scribes, as his powerful memory meant he had the ability to perform all calculations within his own mind. To gather his own information, he had people read aloud to him from scientific articles so he could envision the calculations and memorise the knowledge. Notably, he produced a 775-page textbook on algebra, detailing integral calculus and the moon's motion (Dunham 1999).

4.5 Return to St. Petersburg Academy

Catherine the Great came into power in 1762 and improved the conditions of Russia's politics that had once declined. Due to Euler feeling compelled to leave Berlin, St. Petersburg deemed a beneficial outcome for him. In the year of 1766, Euler returned to the Academy straight into welcome and open arms. He took up residence close to the Academy and was frequently visited by Russian tsarina Catherine II herself (Bradley and E. Sandifer 2007). Euler worked at the Academy until the time of his death. Euler had a strained relationship with Vladimir Grigorievich Orlov, the director of the St. Petersburg Academy causing Euler to eventually resign from his academic jobs. However, he continued to work privately within his research(Bradley and E. Sandifer 2007).

4.6 Later Life & Death

The year 1771 was a chaotic year for Euler, his house caught fire burning down his personal library and a mass of his work. However, his manuscripts were retrieved so his life work was not completely lost (Wilson 2007). Euler became practically blind in 1771, yet he still managed to write a paper on mathematics every week for the whole of 1775 (Dunham 1999). He also published multiple works to which he verbally delivered so young mathematicians could transcribe the long mathematical formulas to paper (Bradley and E. Sandifer 2007). Euler was a mentor to Nikolaus Fuss who married Leonhard Euler's granddaughter Albertine Euler. She was the daughter of Euler's eldest son Johann Albrecht (Bradley and E. Sandifer 2007). Euler's incredible memory was useful during the period in which he lost his sight as he could remember formulas and calculations rather than working it out by hand. Euler produced such a large quantity of work that St. Petersburg Academy struggle to publish everything he had produced, hence, the publishing continued for some time after Euler's death (Bradley and E. Sandifer 2007). In 1773, Euler's wife Katharina died. Three years after, Euler married Katharina's half sister and the marriage lasted until his death in 1783 (Dunham 1999). Euler sadly died September 18th 1783, and was buried in St. Petersburg. His cause of death was a haemorrhage in the brain, leading to a stroke (Dunham 1999; Calinger 2015). He had a total of 45 grandchildren, however only 26 survived after his death (Calinger 2015).

5 Fundamental Topics For The Basel Problem

5.1 Euler's Notation For Functions

> "A function is a rule which gives only one output for every given input"

(Singh 2011)

Euler describes that "A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant qualities" (Euler 1748; Bruce 2013). Euler was the first mathematician to adopt the notation $f(x)$. The idea of the notation is credible, as "f is the first letter of the word 'function' and x normally represents an independent variable" (Singh 2011). He was influenced to make the change by Johann Bernoulli and adapted Bernoulli's *f of x* to $f(x)$. Rüdiger Thiele claimed "Euler gave the function its analytical meaning" (MAA 2019). The meaning of this statement proves the impact Euler's notation had on future mathematics. Euler developed an extensive understanding of functions as his life work went on.

The name 'function' is derived from 'functio', the Latin word that refers to an activity or a performance (MAA 2019). Johann Bernoulli's definition of a function was remarked as "One quantity composed from one or a greater number of quantities" (Calinger 2015). The function shorthand has been continually used and still is in the present day. The notation $f(x)$ was not accepted within mathematics for a long period of time, but noticeably, it is an important concept in all areas of mathematics in the current day. Before the notation of a function was used, mathematicians would use algebraic expressions to describe curves or formulas on its own rather than equating the expression to a variable or symbol (MAA 2019).

Euler was motivated from Leibniz and Bernoulli's work on functions, and split functions into categories, algebraic and transcendental (Bradley and E. Sandifer 2007). Euler advanced his use of functions in "Introductio in analysin infinitorum" by looking at the transcendental functions which are exponential, logarithmic and trigonometric (Bradley and E. Sandifer 2007). He was also able to form polynomials using functions and find inverses of functions. An example of a function and it's inverse he investigated is the exponential function and its inverse which is the logarithm.

$$f(x) = e^x \rightarrow f^{-1}(x) = log_e(x)$$

In Euler's first attempt of the Basel Problem, the expression includes the natural logarithm. Additionally, functions relate closely to the Basel Problem as the start of Euler's method was to observe the expansion of the sine function, $sin(x)$. Euler's independent use of the term influenced many generations. The pure inventiveness of the idea that any curve, system or process can be narrated by this term was a revelation.

5.2 Euler's Number

> "Number whose hyperbolic logarithm $= 1$"

(Euler 1731b)

The number denoted by the letter "e" is an irrational number referred to as "Euler's number". The value is approximately:

$$e \approx 2.71828\ldots$$

The number "e" can be denoted as the sum of the following infinite series:

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$$

(Scheinerman 2017)

Euler specifically did not name the number e after himself, however he did choose the letter "e" to represent the number (Scheinerman 2017). This could be due to Euler's surname or the word "exponential" also beginning with the letter "e". The use of exponential and logarithmic functions aided his discovery within the Basel Problem as he attempted to link the idea of the summation to logarithms before noticing the involvement of π. The irrational number $e \approx 2.71828$... is "the base of hyperbolic logarithms" and can be expressed in various ways (Calinger 2015).

The shorthand for the infinite series and the limit of the expression can be written as:

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

(Wilson 2007)

Euler began estimating the value by computing the number to 23 decimal places by adding up the terms of the infinite series.

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots = 2.718281828459\ldots$$

(Scheinerman 2017)

Euler combined his notation of functions along with the number e and the exponential function was formed. It is believed that Bernoulli presented Euler with the exponential function $f(x) = e^x$ (Bradley and E. Sandifer 2007). The exponential function can be written in the following form as a sum of an infinite series.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

(Dunham 1999)

Euler's work on infinite series began long before his experience with the Basel Problem, and the exponential function led the way to him discovering the theorem that he called his own formula. The development of the exponential function is written as:

1: Euler's Formula

$$e^{i\phi} = cos\phi + isin\phi$$

Firstly, to understand and calculate the above expression, the Maclaurin series is required. The Maclaurin series is a series expansion of a function $f(x)$ with it's derivatives defined at the point $x = 0$. The series sums to a value of n and can be used to find the limits of certain functions. The formula is written by:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

(Singh 2011)

By looking at the Maclaurin series expansion of the exponential function e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

It can now be seen that once the function becomes e^{ix} the substitution $i^2 = -1$ can be applied.

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \dots$$
$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \dots$$

So by substituting in the value for the function $f(x) = e^{ix}$ and then simplifying, the summation can be separated into real and imaginary parts.

$$e^{ix} = (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots) + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots)$$

Now, the functions $f(x) = cos(x)$ and $f(x) = sin(x)$ can be used to show the identity by considering the Maclaurin series expansions.

$$cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$e^{ix} = cos(x) + isin(x) \tag{1}$$

Euler was also able to construct his own identity by substituting in $x = \pi$ into e^{ix}.

$$e^{i\pi} = cos(\pi) + isin(\pi)$$

It is known that the value of $sin(\pi) = 0$ and the value $cos(\pi) = -1$, so the equation can be solved and rearranged to obtain Euler's identity.

$$e^{i\pi} = -1 \iff e^{i\pi} + 1 = 0$$

2: Euler's Identity

$$e^{i\pi} + 1 = 0$$

Additionally, when substituting in e^{-ix}, he was able to form another equation.

$$e^{-ix} = cos(x) - isin(x) \tag{2}$$

By adding equations (1) and (2), the hyperbolic expression for $cos(x)$ can be found.

$$e^{ix} + e^{-ix} = cos(x) + isin(x) + cos(x) - isin(x) = 2cos(x)$$

By rearranging to get $cos(x)$ as the subject:

$$cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

By subtracting equations (1) and (2), the hyperbolic expression for $sin(x)$ can be found.

$$e^{ix} - e^{-ix} = cos(x) + isin(x) - [cos(x) - isin(x)] = 2isin(x)$$

By rearranging to get $sin(x)$ as the subject:

$$sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

When considering Euler's formula, the hyperbolic functions were derived from Euler's influence. Another consequence of Euler's formula is DeMoivre's formula:

3: DeMoivre's Formula

$$e^{in\theta} = (e^{i\theta})^n = (cos(\theta) + isin(\theta))^n = cos(n\theta) + isin(n\theta)$$

5.3 Logarithms

> " I have arrived at the most natural and fruitful concept of logarithm "

(Euler 1748; Bruce 2013)

The inverse function of the exponential function is the logarithmic function.The logarithm was determined by John Napier, with his friend Henry Briggs creating a table for a number of logs at base 10. Euler described the function of $ln(x)$ as:

$$ln(x) = \lim_{n \to \infty} n(x^{1/n} - 1)$$

(Maor 2009)

The natural logarithm is described by the Taylor series expansion at $x = 1$ as the function $ln(x)$ is not defined at the point $x = 0$.

$$ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$$

Euler was able to link his discovery of $e^{ix} = cos(x) + isin(x)$ to the logarithm of negative numbers by using his identity $e^{i\pi} + 1 = 0$. Leibniz believed that $log(-1)$ did not exist and the calculation could not be done. However it was Johann Bernoulli that believed the principle that $log(-x) = log(x)$ (Calinger 2015). Both of the mathematicians were incorrect and Euler stepped up to show that $log(-1) = \pi i$.

By rearranging his identity, -1 can be made the subject of the equation.

$$e^{i\pi} + 1 = 0 \implies e^{i\pi} = -1$$

Then by taking the inverse function of the exponential function, $log(x)$, as $log(e^x) = x$, the following equation shows the result of $log(-1)$ proving both Leibniz and Johann Bernoulli wrong.

$$i\pi = log(-1)$$

The inverse of the exponential function, the logarithms, are used within the Basel Problem in Euler's second attempt at the problem. The natural logarithm is used, and fortunately Euler remembered the value of logarithms along with log tables so he was able to connect the logarithms to the value of the sum of the Basel Problem.

5.4 π

> “ We will use the symbol π for the number. ”

(Euler 1748; Bruce 2013)

π is a Greek letter associated with the approximate value $\pi \approx 3.14159\ldots$. It is used to define the ratio between the circumference of a circle in comparison to it's diameter. To begin with, Euler was disinterested in the mathematical notation, until he discovered the presence of the symbol in various results and equations that did not include circles (C. E. Sandifer 2014). Euler wrote a piece in 1737 discussing the previous findings of π and how the value had been determined (C. E. Sandifer 2014). William Jones began using the π notation in 1706, meaning Euler used the newly surfaced symbol and provided it the publicity it needed (Wilson 2007). Although Euler did not devise π, he did popularise the term as he found the use for it within the Basel Problem and for the now familiar zeta function.

Euler had written volumes of his piece "Introductio in Analysin Infinitorum" in which he calculated π to 127 decimal places in the first volume (Bradley and E. Sandifer 2007). He also wrote about the idea that π was an irrational number as it "cannot be expressed exactly in rational numbers" (Euler 1748). Euler originally used the letter 'p' to denote π before adopting the Greek letter as a mathematic symbol. It was clear to see that Euler was able to connect certain values to the notation of π and recognised when the value was present.

4: Leibniz Formula

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

The formula is found by substituting in $x = 1$ into the formula for the inverse function of $tan(x)$.

$$tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

By substituting in $x = 1$, it is known that $tan^{-1}(1) = \pi/4$

$$tan^{-1}(1) = 1 - \frac{1^3}{3} + \frac{1^5}{5} - \frac{1^7}{7} + \cdots = \frac{\pi}{4}$$

(Euler 1748)

Therefore, the above equation can be rearranged in terms of π.

$$\pi = 4tan^{-1}(1)$$

5.5 Infinite Series

> "Euler must be regarded as the first master of the theory of infinite series"

(Varadarajan 2007)

Euler began looking at infinite series before the Basel Problem, including the p-series where $p = 1$ before successfully attempting $p = 2$.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$$

He devised his own proof to the p-series when $p = 1$ and linked the series to the logarithmic functions.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

Euler began with the Maclaurin series expansion of $ln(1 - x)$.

$$ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$$
$$= -\Big(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots\Big)$$

Now by letting $x = 1$, the series becomes:

$$ln(0) = -\Big(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\Big)$$

Therefore:

$$ln(0) = -\Big(\sum_{n=1}^{\infty} \frac{1}{n}\Big)$$

By rearranging and solving:

$$\sum_{n=1}^{\infty} \frac{1}{n} = -ln(0) = ln\Big(\frac{1}{0}\Big)$$
$$= ln(\infty) = \infty$$

Euler was correct in stating the sum of the series equals infinity, however, $1/0$ is undefined rather than equalling ∞. Euler continued connecting the logarithmic functions with the harmonic series. He put $x = 1/n$ into the Maclaurin series expansion of $ln(1 + x)$.

$$ln\Big(1 - \frac{1}{n}\Big) = ln\Big(\frac{n+1}{n}\Big) = (1/n) - \frac{(1/n)^2}{2} + \frac{(1/n)^3}{3} - \frac{(1/n)^4}{4} + \dots$$
$$= \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \dots$$

By rearranging to get 1/n as the subject:

$$\frac{1}{n} = ln\Big(\frac{n+1}{n}\Big) + \frac{1}{2n^2} - \frac{1}{3n^3} + \frac{1}{4n^4} - \dots$$

Euler interpreted this as 1/n being approximately equal to ln((n+1)/n) for large values of n. He then inferred that harmonic series may be the summation of logarithms. By using a range of values for n, he obtained an array of summations.

For n=1:

$$1 = ln(2) + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots$$

For n=2:

$$\frac{1}{2} = ln\Big(\frac{3}{2}\Big) + \frac{1}{8} - \frac{1}{24} + \frac{1}{64} - \dots$$

$$\vdots$$

For the term n:

$$\frac{1}{n} = ln\Big(\frac{n+1}{n}\Big) + \frac{1}{2n^2} - \frac{1}{3n^3} + \frac{1}{4n^4} - \dots$$

Then by taking the summation of all the expansions:

$$\sum_{k=1}^{n} \frac{1}{k} = \Big[ln(2) + ln\Big(\frac{3}{2}\Big) + ln\Big(\frac{4}{3}\Big) + \dots + ln\Big(\frac{n+1}{n}\Big)\Big] + \frac{1}{2}\Big[1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{n^2}\Big]$$

$$-\frac{1}{3}\Big[1 + \frac{1}{8} + \frac{1}{27} + \dots + \frac{1}{n^3}\Big] + \dots$$

From before, Euler knew that the summation of the bracket containing the logarithms was equal to $ln(1+n)$. However, he did not know the exact value of the remaining terms. Therefore, he summed as many terms possible to find an estimate.

$$\sum_{k=1}^{n} \frac{1}{k} \approx ln(1+n) + 0.577218$$

(Dunham 1999)

This approximate value is now identified as “Euler’s constant", also referred to as γ. The number is found by rearranging the above equation and taking the limit.

5: Euler’s Constant, γ

$$\gamma = \lim_{n\to\infty}\Big(\sum_{k=1}^{n} \frac{1}{k} - ln(n+1)\Big)$$

Euler’s constant, γ was a mathematical breakthrough causing a number of other mathematicians to tackle estimating the value. Lorenzo Mascheroni gave the constant the notation γ and computed an approximation of the value to 32 decimal places (Dunham 1999). Johann Georg von Soldner contradicted Mascheroni by stating that his twentieth decimal place was incorrect. To rectify the issue, Gauss instructed another mathematician to confirm the correct answer was in fact Johann Georg von Soldner. Despite Mascheroni's incorrect approximation, the constant is often referred to as "Euler-Mascheroni's constant" (Dunham 1999). Again, the constant displays another invention of Euler's that caused a ripple effect on mathematics.

6 The Basel Problem

6.1 Previous to Euler

The summation currently known as the Basel Problem troubled multiple mathematicians prior to Euler. The objective of the Basel Problem is to sum the reciprocals of the squares of the natural numbers from 1 to ∞ and find the exact value. The name of the problem actually came from the Bernoulli family, as this was their hometown alongside Euler's (Freiberger 2016). There was vast inspiration for Euler to carry out this equation, as a great deal of mathematicians had failed. So to achieve such a remarkable piece of work was a true success.

Pietro Mengoli, an Italian mathematician was the first familiar mathematician to attack the dilemma in 1644, but addressed the problem in his "Novae quadraturae arithmeticae" in 1650 (Calinger 2015). Mengoli failed to compute the exact value as he only had an estimate from summing a number of terms, so the summation remained unsolved until the following mathematician tackled it. Gottfried Wilhelm Leibniz also found himself short of an answer as he could not determine the overall sum. Leibniz devised calculus and calculated many other infinite series, but even he was left puzzled (Dunham 1999).

John Wallis was up next. In 1656, he identified the problem to be 1.645 which was correct to 3 decimal places. He had written about his discovery in "Arithmetica infinitorum" (Calinger 2015). Again, he could not manage to take the problem any further, so his work stopped there.

Jacob Bernoulli caused the Basel Problem to adopt the name it had. He named the problem in his "Tractus de seriebus infintitis" even though the exact value was still unknown (Calinger 2015). Jacob Bernoulli had studied infinite series intensively. He managed to prove that a harmonic series appears to diverge and calculated the exact sum of plenty of convergent series (Dunham 1999). Bernoulli attempted to find the precise value of the p-series with different values of p.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$$

Bernoulli developed his own proof like Euler did for $p = 1$. They were able to show the p-series is a divergent harmonic series. Next, he let $p = 2$ and so Bernoulli decided to trial the now known Basel Problem and created the following inequality.

It can be seen clearly that:

$$n^2 + n^2 \geq n^2 + n$$

So by simplifying the inequality:

$$2n^2 \geq n^2 + n$$

Then, by taking a factor of n out of the right hand side of the inequality, it becomes:

$$2n^2 \geq n(n+1)$$

Now, by dividing by a factor of 2:

$$n^2 \geq \frac{n(n+1)}{2}$$

Then by taking the inverse of the inequality to get the Basel Problem, the inequality is now:

$$\frac{1}{n^2} \leq \frac{2}{n(n+1)}$$

Therefore, the sum of the inequality for the Basel Problem must be:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \leq \sum_{n=1}^{\infty} \frac{2}{n(n+1)}$$

This can also be written as:

$$= 1 + \frac{1}{4} + \frac{1}{9} + \dots \frac{1}{n^2} + \dots \leq 1 + \frac{1}{3} + \frac{1}{6} + \dots \frac{2}{n(n+1)} + \dots$$

As Bernoulli already knew the value of the telescoping series, $\frac{2}{n(n+1)} = 2$, he could now apply the value to his previous inequality. The following equations show how the exact value of the telescoping series is calculated.

From above, a factor of 2 can be extracted from the summation.

$$\sum_{n=1}^{\infty} \frac{2}{n(n+1)} = 2 \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Then, by using partial fractions, the summation can be written as separate fractions, showing that the summation is a telescoping series.

$$= 2 \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

To expand the series, values of n are substituted into the function, which displays how a telescoping series becomes just two terms as the others will cancel.

$$= 2 \lim_{n \to \infty} \left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \right]$$

The summation is left with two terms, so by letting n tend to ∞ it can be shown that the exact value of the series is 2.

$$= 2 \lim_{n \to \infty} \left[1 - \frac{1}{n+1} \right] = 2 \times \lim_{n \to \infty} \left[1 + 0 \right] = 2 \times 1 = 2$$

By writing the previous inequality, it can be shown that the Basel Problem has an upper bound of value two.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n(n+1)/2} = 2$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2$$

Bernoulli could not develop the problem any further, but was correct with his result that the sum of the series was less than 2. He figured out that as the telescoping series had an exact sum, the Basel Problem must also have an exact sum (Dunham 1999). As Johann Bernoulli's sibling, the Bernoulli family often shared ideas and revelations so it is likely that Jacob presented the question to his brother. As Euler's supervisor and mentor, there is a strong possibility that Euler decided to pursue this complication due to Johann's recommendation and interest (Calinger 2015).

The problem was passed down the generations of the Bernoulli family, and with Euler in connection with the younger Daniel Bernoulli, it could also have been presented to Euler by Daniel. In 1738, Daniel Bernoulli corresponded with Goldbach by writing letters and found the answer to be approximately $8/5$. Daniel stated that if he had noticed the involvement of π then he reckoned he obtained the ability to solve the exact value of the Basel Problem (C. E. Sandifer 2007). Goldbach took this information and chose to compute $(\sum_{n=1}^{\infty} \frac{1}{n^2}) - 1$ (Calinger 2015). Goldbach found the interval for the Basel Problem by finding the upper and lower bounds.

$$\frac{16223}{25200} + 1 < \sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{30197}{46800} + 1$$

$$= 1.6437 < \sum_{n=1}^{\infty} \frac{1}{n^2} < 1.6453$$

(Calinger 2015)

In 1730, James Stirling composed a piece "Methodus differentialis" stating his result to eight decimal places, 1.644034066 (Bradley and E. Sandifer 2007). This was the furthest point any mathematician had got to until Euler. Even Euler's own work provided a starting point as his own understanding of infinite series was extensive.

6.2 Euler's Findings

6: Basel Problem

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

The Basel Problem was not the first time Euler had considered series, but it was possibly his largest finding. Remarkably, it was actually Euler who began the short-hand notation, $\sum$, as a simpler way to write a long or infinite series (Singh 2013). Euler had figured the value to six decimal places in 1729, a year before Stirling found eight decimal places. He found the value to be 1.644924, but this was just the beginning of his search. Euler published his significant discovery in 1740, a few years after the breakthrough was made in 1735 (Suzuki 2009). He began by counting the terms to calculate a rough value to see what region the number remained in (Dunham 1999). Despite using this method, the series converges at a slow rate causing the sum to become more accurate by adding on more terms. To show this, he summed the first ten, hundred and thousand terms.

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{100} \approx 1.54977$$
$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{10,000} \approx 1.63498$$
$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{1,000,000} \approx 1.64393$$

(Dunham 1999)

Euler eventually calculated the value correct to an incredible twenty decimal places. This means that the value changes dramatically when including hundreds of terms on so it can be tedious work. Therefore, Euler took a more rigorous approach when attempting to find his proof. Euler took several approaches when attempting to attain a value as mathematicians such as Daniel Bernoulli were not convinced by his workings.

6.2.1 Theory I

7: Theory I for the Basel Problem

$$\int_0^1 \frac{1}{x}\Big(\int_0^x \frac{1-z^n}{1-z}dz\Big)dx = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2}$$

Euler uses the following sum of the finite geometric series to obtain an estimate for the Basel Problem.

$$\frac{1-x^n}{1-x} = 1 + x + x^2 + \cdots + x^{n-1} \tag{3}$$

(Ed Sandifer 2006; Bruce 2016; Euler 1731a)

By integrating the expression:

$$\int \frac{1-x^n}{1-x}dx = \int (1 + x + x^2 + \cdots + x^{n-1})dx$$
$$\int (1 + x + x^2 + \cdots + x^{n-1})dx = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n}$$

Then, by integrating the expression again:

$$\int \Big(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n}\Big)dx = \frac{x^2}{2} + \frac{x^3}{2 \times 3} + \cdots + \frac{x^{n+1}}{n(n+1)}$$

Euler noticed that the denominators of the expression were not of the form he was hoping for. The denominators are not the square numbers, but instead the product of $n(n+1)$. So he decided to divide the terms by x before integrating for the second time.

$$\int \frac{1}{x}\Big(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n}\Big)dx = \int \Big(1 + \frac{x}{2} + \frac{x^2}{3} + \cdots + \frac{x^{n-1}}{n}\Big)dx$$

$$= x + \frac{x^2}{2 \times 2} + \frac{x^3}{3 \times 3} + \cdots + \frac{x^n}{n \times n}$$

By simplifying:

$$= x + \frac{x^2}{4} + \frac{x^3}{9} + \cdots + \frac{x^n}{n^2}$$

Now, it is clear that when taking the upper limit of the integral as 1 and the lower integral as 0, the Basel Problem is obtained.

$$\int_0^1 \frac{1}{x}\Big(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n}\Big)dx = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2}$$

By substituting equation (3), the Basel Problem is equal to the following double integral.

$$\int_0^1 \frac{1}{x}\Big(\int_0^x \frac{1-z^n}{1-z}dz\Big)dx$$

Then, Euler had to solve the double integral and take the limit of the expression as n tends to infinity. As the integral is difficult to solve for, Euler expanded by terms and added a number of terms to gain an approximate value (Ed Sandifer 2006). Euler arrived at the value 1.644924, and he recognises the value is close to the value $\pi^2/6$. Euler continues with his work on the Basel Problem so he can derive an exact proof rather than an estimate.

6.2.2 Theory II

In a letter to Goldbach in 1730, he constructed the following equation to attempt to solve for the exact value.

8: Theory II for the Basel Problem

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = (ln(2))^2 + \sum_{n=1}^{\infty} \frac{1}{n^2 2^{n-1}}$$

(Calinger 2015)

By computing the Maclaurin series of $ln(1-x)$:

$$ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots$$

The following integral, I, can now be expressed in a variety of ways.

$$I = \int_0^{1/2} -\frac{ln(1-x)}{x}dx$$

By replacing $ln(1-x)$ with the Maclaurin series calculated above:

$$I = \int_0^{1/2} -\frac{(-x - x^2/2 - x^3/3 - \dots)}{x}dx$$

Simplifying:

$$= \int_0^{1/2} \Big(1 + \frac{x}{2} + \frac{x^2}{3} + \dots\Big) dx$$

Integrating:

$$= \Big[x + \frac{x^2}{4} + \frac{x^3}{9} + \dots\Big]_0^{1/2}$$

Finally, substituted in the limits to find the value:

$$= \frac{1}{2} + \frac{(1/2)^2}{4} + \frac{(1/2)^3}{9} + \dots = \sum_{n=1}^{\infty} \frac{(1/2)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{2^n n^2} \tag{4}$$

Euler was able to work out the sum of the Basel Problem by using two versions of the integral I. The second way Euler integrated, is by using the substitution $z = 1 - x$ (Dunham 1999).

As $z = 1 - x$, this means that $dz = -dx$ and the values of the limits of integration change so $z = 1 - (1/2) = 1/2$ and $z = 1 - 0 = 1$. Also, the denominator changes as $z = 1 - x \implies x = 1 - z$.

$$I = \int_0^{1/2} -\frac{ln(1-x)}{x} dx = \int_1^{1/2} \frac{ln(z)}{1-z} dz$$

By recognising that $1/(1-z)$ is the sum of the geometric series $1 + z + z^2 + \dots$, the result can be substituted into the integral to make the integral simpler.

$$= \int_1^{1/2} (1 + z + z^2 + \dots)(ln(z)) dz$$

To make the integration simpler, the substitution z^n can be used to integrate.

$$= \int_1^{1/2} z^n ln(z) dz = \Big[\frac{z^{n+1}}{n+1} ln(z) - \frac{z^{n+1}}{(n+1)^2}\Big]_1^{1/2}$$

Now by substituting back in a number of values for n:

$$= \Big[\Big(zln(z) - z\Big) + \Big(\frac{z^2}{2} ln(z) - \frac{z^2}{4}\Big) + \Big(\frac{z^3}{3} ln(z) - \frac{z^3}{9}\Big) + \dots\Big]_1^{1/2}$$

The result can be factorised:

$$= \Big[ln(z)\Big(z + \frac{z^2}{2} + \frac{z^3}{3} + \dots\Big) - \Big(z + \frac{z^2}{4} + \frac{z^3}{9} + \dots\Big)\Big]_1^{1/2}$$

From observing this expression, it is clear to see that the expansion for $ln(1-z)$ is present, so this can be replaced in the equation. We can also recognise the other expression to be the Basel Problem itself multiplied by z^n, so the notation of the summation can also be replaced.

$$= \Big[ln(z)\Big(-ln(1-z)\Big) - \Big(\sum_{n=1}^{\infty} \frac{z^n}{n^2}\Big)\Big]_1^{1/2}$$

The limits are then substituted into the integral:

$$= \Big[ln(1/2)\Big(-ln(1/2)\Big) - \Big(\sum_{n=1}^{\infty} \frac{(1/2)^n}{n^2}\Big)\Big] - \Big[ln(1)\Big(-ln(0)\Big) - \Big(\sum_{n=1}^{\infty} \frac{1^n}{n^2}\Big)\Big]$$

This expression can then be simplified, but in spite of Euler assuming that $ln(1)ln(0) = 0$, this can be proved in the current day using L'Hôpital's rule than $lim_{z\to 1} ln(z)ln(1-z) = 0$, which confirms his assumption to be correct (Dunham 1999).

$$= -[ln(2)]^2 - \sum_{n=1}^{\infty} \frac{1}{n^2 2^n} + \sum_{n=1}^{\infty} \frac{1}{n^2} \tag{5}$$

Finally, Euler equated (1) and (2) as they are both equal to I and came to the conclusion for the Basel Problem by rearranging:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = -[ln(2)]^2 - \sum_{n=1}^{\infty} \frac{1}{n^2 2^n} + \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\implies \sum_{n=1}^{\infty} \frac{1}{n^2 2^n} = [ln(2)]^2 + 2\sum_{n=1}^{\infty} \frac{1}{n^2 2^n}$$

$$= [ln(2)]^2 + \sum_{n=1}^{\infty} \frac{1}{n^2 2^{n-1}}$$

Due to Euler's extensive memory, he was able to memorise natural logs, although the expression is only able to provide an estimate rather than an exact value. This approach is similar to Theory I, suggesting he perhaps used his previous theory as a starting point and began to alter the derivation. However, this proof was still not the result he was seeking. So the following approach shows the method for connecting the Basel Problem to π.

6.2.3 Theory III

9: Theory III for the Basel Problem

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

This proof displays the derivation of the exact value, so by first inspecting the Maclaurin series expansion for the sine function (Sullivan 2013):

$$sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Now, by dividing through by x, the remaining terms are:

$$\frac{sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots$$

The roots of $sin(x)/x$ are found by letting $sin(x)$ be equivalent to zero. To do this, $sin(x) = sin(n\pi)$ therefore $x = n\pi$ where $n = 1, 2, 3, \dots$

Now the expansion can be rewritten by factorising the expression into a polynomial as the roots are known.

$$\frac{\sin(x)}{x} = \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 + \frac{x}{3\pi}\right)\dots$$

Then by partially expanding:

$$= \Big(1 - \frac{x^2}{\pi^2}\Big)\Big(1 - \frac{x^2}{4\pi^2}\Big)\Big(1 - \frac{x^2}{9\pi^2}\Big)\dots$$

Then by expanding the brackets again and collecting like terms, the calculation becomes:

$$= 1 - \Big(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + \dots\Big)x^2 + \dots$$

Clearly the coefficient of x^2 is equal to the summation of the currently known Basel Problem with a multiple of $1/\pi^2$ so the focus shifts to this sequence.

From the previous expansion of $sin(x)/x$, the coefficient is $-1/3! = -1/6$ so these coefficients can be equated.

$$\begin{aligned} -\frac{1}{6} &= -\Big(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + \dots\Big) \\ &= -\frac{1}{\pi^2}\Big(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots\Big) \\ &= -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} \end{aligned}$$

By rearranging the above equation,

$$-\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = -\frac{1}{6}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \Big(-\frac{1}{6}\Big)\Big(-\pi^2\Big)$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Alternatively written,

$$1 + \frac{1}{4} + \frac{1}{9} + \dots = \frac{\pi^2}{6}$$

6.2.4 Theory IV

10: Theory IV for the Basel Problem

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Euler had many disapproving mathematicians, so he decided to devise another proof to convince his sceptics. This proof separates the odd and even values of n and uses the known values to rearrange and evaluate the Basel Problem.

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

We can use the exact value of a different summation to determine the value of the Basel Problem.

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

By expanding the series by substituting values of n and separating the odd and even n values into brackets and simplifying:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right) + \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \dots\right)$$

$$= \left(1 + \frac{1}{9} + \frac{1}{25} + \dots\right) + \left(\frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \dots\right)$$

A factor of $1/4$ can be extracted from the even values:

$$= \left(1 + \frac{1}{9} + \frac{1}{25} + \dots\right) + \frac{1}{4}\left(1 + \frac{1}{4} + \frac{1}{9} + \dots\right)$$

The equation can now be simplified by rewriting as separate summations.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2}$$

The exact value of the summation of the odd numbers squared is substituted into the expansion:

$$= \frac{\pi^2}{8} + \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2}$$

So now, the equation can be solved by adding the coefficients of the summations and rearranging.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8} + \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{3}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3} \times \frac{\pi^2}{8} = \frac{\pi^2}{6}$$

Euler's studies of the value of π aided his discovery as he was able to recognise the symbol present in this equation. Consequently, Euler spent a further 8 years pursuing an improved proof for his findings (Calinger 2015).

6.3 After Euler's Discovery

> "The eighteenth century mathematics belonged to Euler."

(Singh 2011)

Euler persevered with his advanced mathematics and decided to continue to distinguish exact values of infinite series. Euler extended this by totalling the terms of the series for values of s up to 12 for $\sum_{n=1}^{\infty} 1/n^s$ (Bradley and E. Sandifer 2007). Although, he struggled to sum the series when s is an odd number. It was known that the solution to:

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Also the Basel problem provided the solution:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Now, by attempting to solve:

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

It appears clear that the solution would be of the form π^3/k where k is an integer in the interval 6 to 90 (Dunham 1999). Euler reverted back to his initial method and computed the total of a number of terms and found the result to 9 decimal places. The value 1.202056903 to be approximately equal to $\sum_{n=1}^{\infty} 1/n^3$ meaning the value of k would not be a whole number (Dunham 1999). Euler then produced the formula for the summation of the reciprocals of the cubes of natural numbers.

11: Attempt to solve $\zeta(3)$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \alpha(ln(2))^2 + \beta\frac{\pi^2}{6}ln(2)$$

Where α and β are rational numbers.

(Dunham 1999)

However, Euler could not develop this theory any further, so the victory was left for another mathematician to capture.

Conclusively, Euler confirmed that $\sum_{n=1}^{\infty} 1/(2k)$ will always have the sum π^{2k} multiplied by a rational number. He was able to denote $\zeta(2k)$ by using the Bernoulli numbers.

12: Expression for $\zeta(2k)$ Using the Bernoulli Numbers

$$\zeta(2k) = \frac{(-1)^{k-1} B_{2k} (2\pi)^{2k}}{2(2k)!}$$

For $n \geq 1$ and where B_{2k} are the Bernoulli numbers.

(Calinger 2015)

Centuries later in 1978, Roger Apéry identified the quantity of $\sum_{n=1}^{\infty} 1/n^3$ is indeed an irrational number (Dunham 1999). This was identified purely off of the basis of Euler's work but an exact answer is yet to be found. It is still unknown whether the value is transcendental or not,

Bernard Riemann developed the idea of the Basel Problem, and created the zeta symbol so the problem was now known as $\zeta(2)$ (Calinger 2015). Euler was the inspiration for Riemann's breakthrough with the new Riemann zeta function. The reason the Riemann zeta function was not claimed by Euler was due to the fact he did not acknowledge the ζ symbol itself, because he focused purely on specific values of s rather than developing the function. Therefore, Bernard Riemann was able to identify the function and evolve the function from Euler's form. The zeta function is defined:

13: Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

The following expression is an alternative way to write the zeta function, but it is also know as Euler product.

14: Euler Product

$$\zeta(s) = \prod_{p-prime} \left(1 - \frac{1}{p^s}\right)^{-1}$$

(E. C. Titchmarsh, E. C. T. Titchmarsh, Heath-Brown, et al. 1986)

This can be demonstrated by expanding $\sum_{n=1}^{\infty} 1/n^s$ for an amount of terms.

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{8^s} + \frac{1}{9^s} + \frac{1}{10^s} + \frac{1}{11^s} + \frac{1}{12^s} + \dots$$

Then by separating the denominators into their prime factors:

$$= \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{2^{2s}} + \frac{1}{5^s} + \frac{1}{2^s \times 3^s} + \frac{1}{7^s} + \frac{1}{2^{3s}} + \frac{1}{3^{2s}} + \frac{1}{2^s \times 5^s} + \frac{1}{11^s} + \frac{1}{2^{2s} \times 3^s} + \dots$$

Factorising the expressions:

$$= \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \frac{1}{2^{3s}} + \dots\right)\left(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \frac{1}{3^{3s}} + \dots\right)\left(1 + \frac{1}{5^s} + \frac{1}{5^{2s}} + \frac{1}{5^{3s}} + \dots\right)\dots$$

$$= \prod_{p=primes} \frac{1}{1-\frac{1}{p^s}} = \prod_{p-prime} \left(1-\frac{1}{p^s}\right)^{-1}$$

"The Dirichlet series $\sum_{n=1}^{\infty} n^{-s}$ and the Euler product $\prod_p (1-p^{-s})^{-1}$ converges absolutely in the half plane $\sigma > 1$ and uniformly in each compact subset of this half plane" (Nakamura 2015). The Riemann hypothesis is an extension of the Riemann zeta function meaning that it is also derived from the Basel Problem. The Riemann hypothesis was produced also by Bernard Riemann in 1859 and is the idea that all non-trivial zeros may lie on the critical line $\sigma = 1/2$ (Nakamura 2015). The hypothesis states that every non-trivial zero's real part is equal to $1/2$. The hypothesis can also be written:

15: Riemann Hypothesis

$$\zeta(s) \neq 0$$

When $1/2 < \sigma < 1$

Hadamard and de la Vallée-Poussin verified that $\zeta(1+it) \neq 0 \ \forall t \in \mathbb{R}$ in 1896 (Nakamura 2015). The solution to the Riemann hypotheisis is arguably the $21st$ century Basel Problem as mathematics is still awaiting it's proof. The confirmation of the Riemann hypothesis will display prime number distributions which will lead to other discoveries.

7 Conclusion

To summarise this project, Euler's solution to the Basel Problem was exceedingly successful and his mathematics allowed further development. As displayed, his work on the Basel Problem was influenced by Pietro Mengoli, Gottfried Wilheim Leibniz, Jacob Bernoulli and a number of other powerful mathematicians. Euler's breakthrough with the summation allowed for a general expression to be found for the condition $\zeta(2n)$, which determined the exact sum for plenty of expressions.

In conclusion, Euler was directed in his career through political and economic issues. His relocation's throughout his life provided him with skills and connections with people that influenced his mathematics. The Bernoulli family were a strong force on Euler's discovery as Johann mentored Euler, supplying guidance and possibly exposing Euler to the Basel Problem. It is known that Euler developed Johann's use of *f of x* and it is believed that Johann presented Euler with the exponential function e^x. Daniel played an important role in Euler's life and was not afraid to become Euler's constructive critic. Daniel's disbelief in the Basel Problem gave Euler the motivation to continue his investigation to prove he had found the correct exact answer. Additionally, Daniel provided Euler with a support system, as he could discuss theories freely with him. The financial state in Basel aided Euler's career otherwise he may never have moved to St. Petersburg and may never have successfully discovered the Basel Problem. Even though Euler was an intelligent man, his life story encouraged him within mathematics and is a crucial section of this project.

To extend this project, the Riemann hypothesis could be investigated in further depth. Due to this hypothesis not yet being proved, the investigation would become centred around the hypothesis compared to the Basel Problem. Hence, the decision to summarise the hypothesis was made to concentrate on Euler's findings on the Basel Problem. The γ notation could also have been developed as it was an critical finding of Euler's, however, it is not relevant to the proofs of the Basel Problem.

Overall, the St. Petersburg Academy and the Berlin Academy also encouraged his findings as he exercised his mathematics through his entries to the academies journals. His friendship with the Bernoulli's inspired him to persevere with the Basel Problem and with mathematics due to their shared passion of the subject. Although it was Euler's connection of π with the Basel Problem and that was his greatest success. Without this link, his proofs would not have been possible.

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