Problems on Chapter 1

1. How many entries does a 512 by 1024 matrix have?

2. Does a symmetric matrix have to be a square matrix?

3. Let **A** be a symmetric matrix. Show that for a natural number n we have $\left(\mathbf{A}^T\mathbf{A}\right)^n = \mathbf{A}^{2n}$.

4. Determine **AB** if possible for the following matrices:

   (a) $\mathbf{A} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 5 & 6 \end{pmatrix}$

   (c) $\mathbf{A} = \begin{pmatrix} 1 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ (d) $\mathbf{A} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 2 \end{pmatrix}$

5. A curve C is given by the equations $\begin{pmatrix} x & y & z \end{pmatrix}^T$ where

$$\begin{aligned} x &= a_1t^3 + b_1t^2 + c_1t + d_1 \\ y &= a_2t^3 + b_2t^2 + c_2t + d_2 \\ z &= a_3t^3 + b_3t^2 + c_3t + d_3 \end{aligned}$$

   Write these equations in matrix form.

6. Let $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$.

Determine $\sum_{j=1}^{2} \mathrm{col}_j(\mathbf{A})\mathrm{row}_j(\mathbf{B})$ where $\mathrm{col}_j(\mathbf{A})$ is the jth column of **A** and $\mathrm{row}_j(\mathbf{B})$ is the jth row of matrix **B**. *What do you notice about your result?*

7. Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$. Find the second row of $\mathbf{AB}$ without evaluating all of $\mathbf{AB}$.

8. Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & 1 & 1 & 2 & 4 & 5 & 6 \\ 3 & 7 & 6 & 9 & 1 & 3 & 5 \\ 2 & 5 & 7 & 8 & 6 & 1 & 3 \\ 1 & 3 & 5 & 3 & 4 & 6 & 8 \end{pmatrix}$. Determine the fourth column of $\mathbf{AB}$ without evaluating all the entries of **AB**.

9. A *non-negative* matrix is a matrix in which all its entries are non-negative. A *stochastic matrix* is a square matrix in which the sum of each of the rows is equal to one.
   Give examples of non-negative stochastic matrices.

10. Consider the linear system $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ 4 & 1 & 3 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Solve this linear system for
    (i) $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (ii) $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (iii) $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

11. Find matrices **A** and **B** such that $\mathbf{AB} = \mathbf{BA}$ where **A** and **B** are not the identity or the zero matrices.
    For your chosen matrices determine $(\mathbf{A} + \mathbf{B})^2$.

12. Find two different non-invertible (singular) matrices, **A** and **B**, such that $\mathbf{A} + \mathbf{B}$ is invertible.

13. Determine values of *k* so that the following linear system has:

$$\begin{array}{ccccccc} x & + & y & + & kz & = & 1 \\ 2x & + & ky & - & 3z & = & 2 \\ kx & + & 2y & + & 6z & = & 0 \end{array}$$

(a) no solution (b) unique solution (c) infinite number of solutions

14. Consider the following linear system $\mathbf{Ax} = \mathbf{O}$:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 2 \\ 5 & 1 & 3 & 1 \\ 6 & 1 & -4 & 4 \\ 7 & 1 & 1 & 2 \\ 4 & 1 & 1 & 3 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

The reduced row echelon form of matrix **A** is given by **R**:

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Solve $\mathbf{Ax} = \mathbf{O}$.

15. Find the inverse of the following matrices given that $a \neq 0,\ b \neq 0$ and $c \neq 0$:

(a) $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ (c) $\mathbf{C} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$

16. Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix}$ where $a,\ b$ and c are non-zero numbers.

Evaluate (i) $\mathbf{AB}$ (ii) $(\mathbf{AB})^{-1}$ (iii) $\mathbf{BA}$ (iv) $(\mathbf{BA})^{-1}$

17. Find the inverse of matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

18. Find non-zero matrices **A** and **B** such that $\mathbf{AB} = \mathbf{O}$ where **O** is the zero matrix.

19. Let $\mathbf{A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Determine $\mathbf{A}^{2013}$.

20. Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \\ 9 & 10 \end{pmatrix}$. Find $\mathbf{A}^T\mathbf{A}$.

21. Let $\mathbf{E} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Write the matrix $\mathbf{A} = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$ as a polynomial in the matrix **E**.

22. Find a matrix **A** which satisfies $\mathbf{A}^n - \mathbf{A} = \mathbf{O}$.

23. Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Evaluate **ADB**.

24. Let $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} 1 & 2 & -6 \\ 2 & 2 & 1 \\ -6 & 1 & 3 \end{pmatrix}$. Determine $\mathbf{x}^T\mathbf{A}\mathbf{x}$. (This $\mathbf{x}^T\mathbf{A}\mathbf{x}$ is called *quadratic form*.)

25. Show the following result is *false*:
Let **A** be a real 2 by 2 matrix. If $\mathbf{A}^2 = \mathbf{O}$ then $\mathbf{A} = \mathbf{O}$.

26. Let **B** be an invertible matrix such that $\mathbf{AB} = \mathbf{O}$. Prove that $\mathbf{A} = \mathbf{O}$.

27. Find the inverse of $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$. Hence or otherwise solve the linear system:

$$\mathbf{Ax} = \begin{pmatrix} 6 & 9 & 6 \end{pmatrix}^T \text{ where } \mathbf{x} = \begin{pmatrix} x & y & z \end{pmatrix}^T$$

28. Let $\mathbf{A} = \begin{pmatrix} 8 & 10 \\ -6 & -8 \end{pmatrix}$.
(a) Find $\mathbf{A}^2$, $\mathbf{A}^3$.
(b) Show that

$$\mathbf{A}^n = \begin{cases} 2^n \mathbf{I} & \text{if } n \text{ is even} \\ 2^n \begin{pmatrix} 3 & 5 \\ -3 & -4 \end{pmatrix} & \text{if } n \text{ is odd} \end{cases}$$

where **I** is the identity matrix.

29. An anti-symmetric or skew symmetric matrix is a square matrix such that $\mathbf{A} = -\mathbf{A}^T$. Prove that if matrices **A** and **B** are anti-symmetric matrices then $\mathbf{AB} - \mathbf{BA}$ is also an anti-symmetric matrix.

30. Prove that if $\mathbf{AB}$ is invertible then $\mathbf{A}$ is invertible and **B** is invertible.

31. Show that $trace(\mathbf{AB}) = trace(\mathbf{BA})$ where **A** and **B** are square matrices.

32. A square matrix **A** is called a nilpotent matrix if $\mathbf{A}^k = \mathbf{O}$ for some positive integer *k*.
(a) Give an example of a non-zero nilpotent matrix.
[Hint: Consider a triangular matrix.]
(b) Prove that if matrices **A** and **B** are nilpotent then **AB** is nilpotent.
(c) Let matrices **A** and **B** be nilpotent. Is the matrix $\mathbf{A} + \mathbf{B}$ nilpotent?

33. The exponential of a matrix **A** which is denoted by $e^{\mathbf{A}}$ is defined

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \frac{1}{3!}\mathbf{A}^3 + \cdots + \frac{1}{n!}\mathbf{A}^n + \cdots$$

Determine $e^{\mathbf{A}}$ for $\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$.

34. Prove that if **A** is invertible then $\mathbf{A}^n$ (*n* is a natural number) is invertible.

35. Let **A** and **B** be *n* by *n* matrices such that $\mathbf{A}^2 = \mathbf{B}^2 = \mathbf{I}$ and $\mathbf{AB} + \mathbf{BA} = \mathbf{O}$. Determine $(\mathbf{A} + \mathbf{B})^{-1}$.

36. Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 0 \\ 3 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & 64 \\ 0 & 0 & 64 & -48 \\ 0 & 64 & -48 & 4 \\ 64 & -48 & 4 & 5 \end{pmatrix}$.

(i) Show that $\mathbf{AB} = 256\mathbf{I}$.

(ii) Find the inverse of matrix **A**.

(iii) Find the inverse of matrix **B**.

37. Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \end{pmatrix}$. Determine $\mathbf{A}(\mathbf{BC})$.

38. Let **A** be an *n* by 1 matrix. Prove that $\mathbf{AA}^T$ is an *n* by *n* symmetric matrix.

39. Let **A** and **B** be symmetric matrices. Prove that **AB** may *not* symmetric.

40. Let **A** be an *n* by *n* matrix such that the entries $a_{ii} \neq -1$ for $i = 1, 2, \cdots, n$. Show that the matrix $\mathbf{A} + \mathbf{I}$ in invertible.

41. Let $\mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$ where $a \neq 0$. Prove that for any natural number *n* we have

$$\mathbf{A}^n = \begin{pmatrix} a^n & 0 \\ 0 & 1/a^n \end{pmatrix}$$

Determine the inverse of $\mathbf{A}^n$.

42. Show that in general $(\mathbf{A} + \mathbf{B})^{-1} \neq \mathbf{A}^{-1} + \mathbf{B}^{-1}$.

43. Show that if matrix $\mathbf{A}^2$ is invertible then so is **A**.

44. Suppose the matrix **A** satisfies the following equation:

$$\mathbf{A}^3 + \mathbf{A}^2 + \mathbf{A} - 10\mathbf{I} = \mathbf{O}$$

Show that matrix **A** is invertible (non-singular) and find an expression for the inverse of **A**.

45. Find a 2 by 3 matrix **A** such that $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

46. Let **A** and **B** be same size invertible matrices. Find the first error, if any, in the following derivation and give reasons for your answer:

$$(\mathbf{AB})(\mathbf{BA})^{-1} = \mathbf{A}\left(\mathbf{BB}^{-1}\right)\mathbf{A}^{-1}$$
$$= \mathbf{AA}^{-1} = \mathbf{I}$$

47. Let $\mathbf{I} + \mathbf{A}$ be an invertible matrix. Find the first error, if any, in the following derivation:

$$(\mathbf{I} + \mathbf{A})\mathbf{X} = \mathbf{B}$$
$$\mathbf{X} = (\mathbf{I} + \mathbf{A})^{-1}\mathbf{B}$$
$$= \mathbf{IB} + \mathbf{A}^{-1}\mathbf{B}$$

48. Show that the vector $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is a *linear combination* of $\mathbf{v} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$.

49. Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 1 & -1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{pmatrix}$.

(i) Show that $\mathbf{AB} = \mathbf{I}$.

(ii) Determine $\left(\mathbf{A}^T\right)^{-1}$.

50. Show that if $\mathbf{x}_1,\ \mathbf{x}_2,\ \cdots,\ \mathbf{x}_n$ are solutions to the linear system $\mathbf{Ax} = \mathbf{0}$ then the linear combination $k_1\mathbf{x}_1 + k\ \mathbf{x}_2 + \cdots + k_n\mathbf{x}_n$ (where k's are non-zero scalars) is also a solution to $\mathbf{Ax} = \mathbf{0}$.

51. Let **A** be an invertible matrix. Prove that

$$\mathbf{I} - \mathbf{A}\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T = \mathbf{O} \text{ where } \mathbf{I} \text{ is the identity matrix}$$

Find an expression for $\mathbf{A}^{-1}$.

52. Let $\mathbf{I}-\mathbf{A}$ be an invertible matrix. Find the first error in the following derivation and give reason for your answer:

$$\begin{aligned} & \mathbf{X}=\mathbf{AX}+\mathbf{Y} \\ \Rightarrow \quad & \mathbf{X}-\mathbf{AX}=\mathbf{Y} \\ \Rightarrow \quad & \mathbf{X}(\mathbf{I}-\mathbf{A})=\mathbf{Y} \\ \Rightarrow \quad & \mathbf{X}=(\mathbf{I}-\mathbf{A})^{-1}\mathbf{Y} \end{aligned}$$

53. Find the first error, if any, of the following derivation(assume all the matrices are invertible):

$$(\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_n)^{-1}(\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_n)=\mathbf{I}$$

$$(\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_n)^{-1}(\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_n)(\mathbf{A}_n^{-1}\mathbf{A}_{n-1}^{-1}\cdots\mathbf{A}_2^{-1}\mathbf{A}_1^{-1})=\mathbf{A}_n^{-1}\mathbf{A}_{n-1}^{-1}\cdots\mathbf{A}_2^{-1}\mathbf{A}_1^{-1}$$

54. Let **P** be an invertible matrix.

(i) Prove that if $\mathbf{A}=\mathbf{PDP}^{-1}$ then $\mathbf{A}^m=\mathbf{PD}^m\mathbf{P}^{-1}$ where *m* is a natural number.

(ii) Let $\mathbf{D}=\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Show that $\mathbf{D}^m=\begin{pmatrix} a^m & 0 \\ 0 & b^m \end{pmatrix}$ where *m* is a natural number.

(iii) Let $\mathbf{A}=\begin{pmatrix} 1 & 2 \\ -9 & -8 \end{pmatrix}$, $\mathbf{D}=\begin{pmatrix} -5 & 0 \\ 0 & -2 \end{pmatrix}$ and $\mathbf{P}=\begin{pmatrix} -1 & -2 \\ 3 & 3 \end{pmatrix}$. Show that $\mathbf{PDP}^{-1}=\mathbf{A}$ and by using the result of part (i) determine $\mathbf{A}^8$.

55. Consider the linear system $\mathbf{Ax}=\mathbf{b}$ where matrix **A** is an *n* by *k* matrix, **x** and **b** are *n* by 1 column vectors. Decide whether the following statements are true. If they are true provide a proof otherwise give a counter example:

(a) If $\mathbf{b}=\mathbf{0}$ then the linear system $\mathbf{Ax}=\mathbf{b}$ has a solution.

(b) If $\mathbf{b}\neq\mathbf{0}$ and $n\neq k$ then the linear system $\mathbf{Ax}=\mathbf{b}$ does *not* have a solution.

(c) If $n>k$ then the linear system $\mathbf{Ax}=\mathbf{b}$ has an infinite number of solutions.

(d) If $n=k$ and the linear system has a solution then it is unique.

(e) If $n<k$ and the linear system has a solution then there are an infinite number of solutions.

56. (Method of least squares.) In engineering and science it is often the case that we need to find a polynomial equation which best fits some experimental data. Suppose from such an experiment we get the following data points (vectors):

$$(1,\ 1.7),\ (2,\ 2.6),\ (3,\ 4),\ (4,\ 5)$$

Assume we are looking for a straight line $y=mx+c$ approximation to this data. By substituting these data values we get the following equations:

$$m+c=1.7,\ 2m+c=2.6,\ 3m+c=4,\ 4m+c=5$$

(i) Write these equations in matrix form $\mathbf{Ax}=\mathbf{b}$ where **A** is the coefficient matrix and $\mathbf{x}=\begin{pmatrix} m \\ c \end{pmatrix}$ is the unknown column vector and **b** is the 4 by 1 column vector.

(ii) Determine $\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T$.

(iii) The least squares solution is given by evaluating

$$\mathbf{x}=\left[\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\right]\mathbf{b}$$

Determine **x**.

(iv) Draw the line $y=mx+c$ for the *m* and *c* values found in part(iii) and the given data points on the same graph.

57. (Cryptography). A simple way of encoding messages is to represent each letter of the alphabet be its position in the alphabet and then add 3 to this. For example we can create the table:

Alphabet	A	B	C	D	...	W	X	Y	Z	
Position	1	2	3	4	...	23	24	25	26	27
Position $+3$	4	5	6	7	...	26	27	28	29	30

Table 1

The last column represents space and we nominate this by a value of $27+3=30$.

Encode the message 'Shall we meet' by using the matrix $\mathbf{A}=\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 3 & 3 & 4 \end{pmatrix}$ and 3 by 1 column vectors. Add spaces in order to complete a 3 by 1 vector.

Find the decoding matrix.

58. Let **A** and **B** be matrices such that $\mathbf{AB}+\mathbf{BA}=\mathbf{O}$. Prove that

$$\mathbf{A}^2\mathbf{B}^3=\mathbf{B}^3\mathbf{A}^2$$

59. Let $\mathbf{D}=\begin{pmatrix} e^{2x} & 0 \\ 0 & e^{x} \end{pmatrix}$ be a diagonal matrix. Prove that $\mathbf{D}^n=\begin{pmatrix} e^{2nx} & 0 \\ 0 & e^{nx} \end{pmatrix}$ where *n* is a natural number.

60. Consider the following magic square:

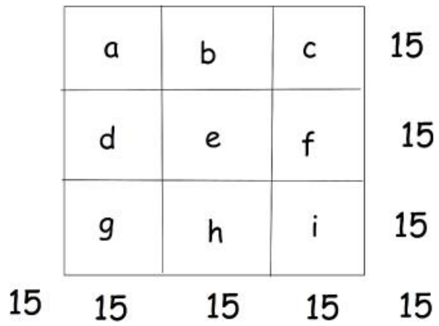

Figure 1

Each row, column and diagonal sum to 15 in this magic square. By writing 8 linear equations and using any maths software find the unknowns in the magic square.

Brief Solutions to Problems on Chapter 1

1. 524288
2. Yes

4. (a) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 7 & 14 & 21 & 28 \end{pmatrix}$ (b) $\begin{pmatrix} 5 & 6 \\ 10 & 12 \\ 15 & 18 \end{pmatrix}$ (c) 11 (d) $\begin{pmatrix} 3 & 6 \\ 4 & 8 \end{pmatrix}$

5. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{pmatrix} \begin{pmatrix} t^3 \\ t^2 \\ t \\ 1 \end{pmatrix}$

6. $\begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$

7. $\begin{pmatrix} 49 & 64 \end{pmatrix}$

8. $\begin{pmatrix} 56 \\ 144 \\ 232 \end{pmatrix}$

9. $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$

10. (i) $s\begin{pmatrix} -1/2 \\ -1 \\ 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1/2 \\ -1 \\ 1 \end{pmatrix}$ (iii) $\begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1/2 \\ -1 \\ 1 \end{pmatrix}$

12. $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is an example.

13. (a) $k = 1,\ 2,\ -3$ (b) $k \neq 1,\ 2,\ -3$ (c) Cannot have infinite number of solutions

14. $x_1 = x_2 = x_3 = x_4 = 0$

15. (a), (b) and (c) are given by

$$\mathbf{A}^{-1}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1/c\end{pmatrix},\ \mathbf{B}^{-1}=\begin{pmatrix}1&0&0\\0&1/b&0\\0&0&1/c\end{pmatrix}\text{ and }\mathbf{C}^{-1}=\begin{pmatrix}1/a&0&0\\0&1/b&0\\0&0&1/c\end{pmatrix}$$

16. (i) $\begin{pmatrix}1&0&0\\a&1&0\\b&c&1\end{pmatrix}$ (ii) $\begin{pmatrix}1&0&0\\-a&1&0\\ac-b&-c&1\end{pmatrix}$ (iii) $\begin{pmatrix}1&0&0\\a&1&0\\b+ca&c&1\end{pmatrix}$ (iv) $\begin{pmatrix}1&0&0\\-a&1&0\\-b&-c&1\end{pmatrix}$

17. No inverse.

18. $\mathbf{A}=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}0&0\\0&1\end{pmatrix}$

19. **A**

20. $\begin{pmatrix}165&190\\190&220\end{pmatrix}$

21. $a\mathbf{E}^3+b\mathbf{E}^2+c\mathbf{E}$

22. **I**

23. 36

24. $x^2+4xy+2y^2+2yz+3z^2-12xz$

27. $\dfrac{1}{24}\begin{pmatrix}24&-12&-2\\0&6&-5\\0&0&4\end{pmatrix}$ and $x=y=z=1$

28. (a) $\mathbf{A}^2=\begin{pmatrix}4&0\\0&4\end{pmatrix}$ and $\mathbf{A}^3=8\begin{pmatrix}3&5\\-3&-4\end{pmatrix}$

32. (a) $\begin{pmatrix}0&1\\0&0\end{pmatrix}$

33. $\begin{pmatrix}1&1&3\\0&1&3\\0&0&1\end{pmatrix}$

35. $\dfrac{1}{2}(\mathbf{A}+\mathbf{B})$

36. (ii) $\mathbf{A}^{-1}=\dfrac{1}{256}\mathbf{B}$ (iii) $\mathbf{B}^{-1}=\dfrac{1}{256}\mathbf{A}$

37. $\begin{pmatrix}29&38&47&56&65\\71&92&113&134&155\end{pmatrix}$

39. $\mathbf{A}=\begin{pmatrix}1&2\\2&3\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}1&0\\0&4\end{pmatrix}$

41. $\left(\mathbf{A}^n\right)^{-1}=\begin{pmatrix}1/a^n&0\\0&a^n\end{pmatrix}$

44. $\mathbf{A}^{-1} = \frac{1}{10}\left(\mathbf{A}^2 + \mathbf{A} + \mathbf{I}\right)$

45. $\mathbf{A} = \begin{pmatrix} -2 & 1 & 1 \\ 9 & -1 & -1 \end{pmatrix}$ (This **A** is not unique.)

48. $2\mathbf{v} - \mathbf{w} = \mathbf{u}$.

49. $\begin{pmatrix} 0 & 2 & -1 \\ 1 & -2 & 1 \\ -1 & -1 & 1 \end{pmatrix}$

51. $\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T$

54. $\begin{pmatrix} -390113 & -260246 \\ 1171107 & 780994 \end{pmatrix}$

55. (a) and (e) True (b), (c) and (d) False

56. (i) $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1.7 \\ 2.6 \\ 4 \\ 5 \end{pmatrix}$ (ii) $\frac{1}{20}\begin{pmatrix} -6 & -2 & 2 & 6 \\ 20 & 10 & 0 & -10 \end{pmatrix}$ (iii) $\begin{pmatrix} 1.13 \\ 0.50 \end{pmatrix}$

57. $37,\ 15,\ 115,\ 60,\ 45,\ 210,\ 64,\ 38,\ 222,\ 32,\ 16,\ 104,\ 77,\ 54$ and 258.

$$\begin{pmatrix} 1 & -1 & 0 \\ 3 & 1 & -1 \\ -3 & 0 & 1 \end{pmatrix}$$