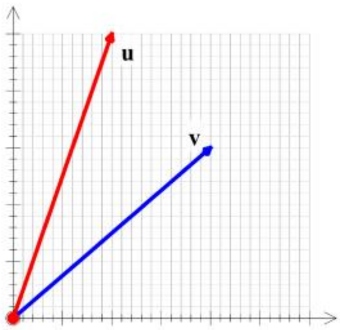
Problems on Chapter 2

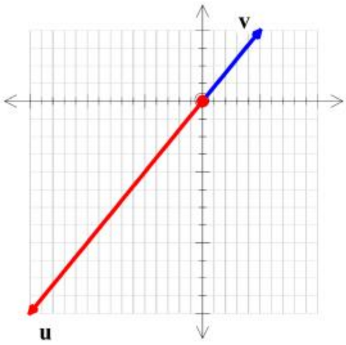
1. Decide whether the following vectors are linearly independent:

$$\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} -3 \\ -6 \\ 9 \end{pmatrix}$$

2. Determine whether the following vectors **u** and **v** are linearly independent:



(a)



(b)

3. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots, \mathbf{v}_n\}$ be a set of vectors in $\mathbb{R}^n$. Prove that if

$$\mathbf{A} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix}$$

has *n* leading columns in reduced row echelon form then the set *S* is a basis for $\mathbb{R}^n$.

4. Show that the vectors $\mathbf{u} = \begin{pmatrix} 2 & -3 & 1 \end{pmatrix}^T$, $\mathbf{v} = \begin{pmatrix} 0 & 1 & 2 \end{pmatrix}^T$, $\mathbf{w} = \begin{pmatrix} 1 & 1 & -2 \end{pmatrix}^T$ form a basis (axes) for $\mathbb{R}^3$, and find the vector representing $\begin{pmatrix} a & b & c \end{pmatrix}^T$ with respect to this basis.

5. Let non-zero vectors **u** and **v** in $\mathbb{R}^n$ be parallel. Show that

$\cos(¯) = \pm 1$ where ¯ is the angle between **u** and **v**

Comment on the value of ¯ .

6. The correlation coefficient *r* between two vectors is given by the cosine of the angle between the vectors. There is a positive (negative) correlation between vectors **u** and **v** if $\mathbf{u} \cdot \mathbf{v} > 0$ ($\mathbf{u} \cdot \mathbf{v} < 0$). If $\mathbf{u} \cdot \mathbf{v} = 0$ then there is *no* correlation between **u** and **v**.

Let $\mathbf{h} = \begin{pmatrix} 1.7 \\ 1.8 \\ 1.9 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 75 \\ 70 \\ 100 \end{pmatrix}$ represent the heights and weights of three people.

Determine the correlation coefficient *r* for these vectors **h** and **w**.

7. Let $\mathbf{u} = \begin{pmatrix} \sin(A) \\ \cos(A) \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} \cos(B) \\ \sin(B) \end{pmatrix}$. Show that

$$\mathbf{u} \cdot \mathbf{v} = \sin(A+B)$$

8. Let $S = Span\left\{\mathbf{u} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}^T\right\}$. Write down the unit vector $\hat{\mathbf{u}}$. Sketch the space *S* and label the unit vector $\hat{\mathbf{u}}$ in this space.

9. Let $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$. Show that $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is an orthogonal basis for $\mathbb{R}^3$. Write $\mathbf{x} = \begin{pmatrix} 7 \\ 1 \\ 9 \end{pmatrix}$ as a linear combination of $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$.

10. You may use Matlab for this question.
    The following set of vectors span $\mathbb{R}^3$:

$$\left\{ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix},\ \mathbf{v}_3 = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix},\ \mathbf{v}_4 = \begin{pmatrix} 10 \\ 11 \\ 11 \end{pmatrix},\ \mathbf{v}_5 = \begin{pmatrix} 13 \\ 14 \\ 15 \end{pmatrix} \right\}$$

    Determine a subset of these which form a basis for $\mathbb{R}^3$.

11. Determine a unit vector **u** which is orthogonal to both

$$\mathbf{v} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}^T \text{ and } \mathbf{w} = \begin{pmatrix} 4 & 5 & 6 \end{pmatrix}^T$$

    Show that $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is a basis (axes) for $\mathbb{R}^3$.

12. Let **u** and **v** be vectors in $\mathbb{R}^n$. Show that

$$d(\mathbf{u}+\mathbf{v},\quad \mathbf{u}-\mathbf{v}) = 2\|\mathbf{v}\| \text{ where } d \text{ is the distance function}$$

    [The distance between vectors $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ is twice the length of the vector **v**.]

13. Show that the following result is *false*:
    Let **u** and **v** be vectors in $\mathbb{R}^n$ and *k* be a scalar then

$$(\mathbf{u}+k\mathbf{v}) \cdot (\mathbf{u}+k\mathbf{v}) = \|\mathbf{u}\|^2 + 2k(\mathbf{u} \cdot \mathbf{v}) + k\|\mathbf{v}\|^2$$

14. Let **u** and **v** be unit vectors in $\mathbb{R}^n$ and *k* and *c* be scalars. Show that

$$(k\mathbf{u}+c\mathbf{v}) \cdot (k\mathbf{u}-c\mathbf{v}) = k^2 - c^2$$

15. Determine whether the vector **b** is in the space spanned by the columns of matrix **A**:

(a) $\mathbf{A}=\begin{pmatrix}1 & 1 & 2\\ 0 & 3 & 1\\ 5 & -7 & 2\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}15\\ 11\\ 11\end{pmatrix}$ (b) $\mathbf{A}=\begin{pmatrix}1 & 1 & 5\\ 0 & 3 & 9\\ 5 & -7 & -11\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}$

16. Let $\mathbf{x}=\begin{pmatrix}x_1 & x_2 & x_3 & x_4\end{pmatrix}^T$ be a vector in $\mathbb{R}^4$ and $\mathbf{e}_1=\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}^T$ and $\mathbf{e}_2=\begin{pmatrix}0 & 1 & 0 & 0\end{pmatrix}^T$ be the unit vectors. For which vectors **x** is the set $\{\mathbf{x},\ \mathbf{x}+\mathbf{e}_1,\ \mathbf{x}+\mathbf{e}_2\}$ linearly independent.

17. Find the value(s) of the scalar *k* so that the following set of vectors are linearly independent:

$$\left\{\begin{pmatrix}k\\ 2\\ 1\end{pmatrix}, \begin{pmatrix}2\\ k\\ 4\end{pmatrix}, \begin{pmatrix}k\\ 12\\ 6\end{pmatrix}\right\}$$

18. Find the value(s) of the scalar *k* so that the following set of vectors are orthogonal:

$$S=\left\{\begin{pmatrix}k\\ 3\\ 2\end{pmatrix}, \begin{pmatrix}k\\ k\\ 1\end{pmatrix}\right\}$$

Find a third vector which is orthogonal to the two vectors in *S* for your *k* values.

19. Let $\mathbf{u}=\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}1\\ 0\\ 3\end{pmatrix}$.

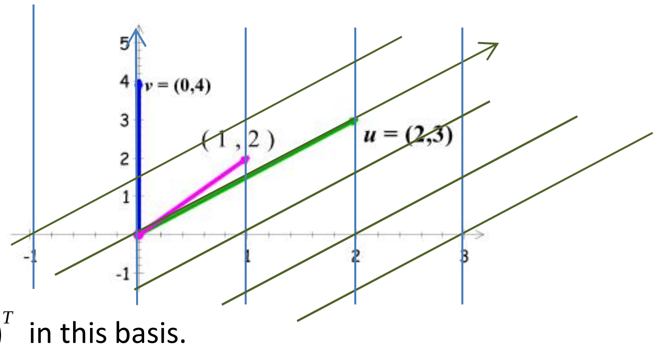
(a) Determine the dependency of **u** and **v**.

(b) Find the space spanned by $\{\mathbf{u},\ \mathbf{v}\}$.

(c) Write down a vector **w** such that $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ form a basis for $\mathbb{R}^3$.

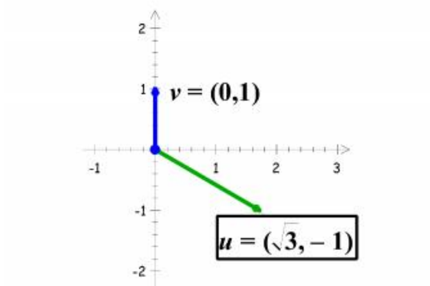
20. Show that $B=\left\{\mathbf{u}=\begin{pmatrix}1\\ -1\end{pmatrix},\ \mathbf{v}=\begin{pmatrix}-1\\ -1\end{pmatrix}\right\}$ is a basis for $\mathbb{R}^2$. Sketch this basis and write the vector $\mathbf{w}=\begin{pmatrix}-2\\ -3\end{pmatrix}$ in this basis.

21. Show that the following vectors $B=\{\mathbf{u},\ \mathbf{v}\}$ are a basis for $\mathbb{R}^2$:



Write the vector $(1 \quad 2)^T$ in this basis.

22. The vectors **u** and **v** show two edges of floor tiling:



Determine the angle between vectors **u** and **v**. Show that $B = \{\mathbf{u},\ \mathbf{v}\}$ is a basis for $\mathbb{R}^2$ which represents the floor. Write the vector $(1 \quad 1)^T$ in this basis.

23. An important application of vectors in $\mathbb{R}^n$ is support vector machines; we are interested in the shortest distance between hyperplanes and vectors – see section B of chapter 2. [The shortest distance from a vector **u** to any point on the hyperplane $\mathbf{v}\cdot\mathbf{x}+c=0$ where $\mathbf{x}=(x \quad y \quad \cdots)^T$ in *n* space can be shown to equal, $\dfrac{|\mathbf{u}\cdot\mathbf{v}+c|}{\|\mathbf{v}\|}$.]

Find the shortest distance between the following vectors and hyperplanes or lines and make a sketch in each case:

(a) Line $y=2x+1$ and the vector $\mathbf{u}=(3 \quad 4)^T$.

(b) The plane $2x+y-5z=5$ and the vector $\mathbf{u}=(1 \quad 2 \quad 3)^T$.

24. Another application of vectors is in information retrieval because vectors can be used to represent documents. A document and a query term can be represented by vectors **d** and **q** respectively. The angle between these vectors **d** and **q** tells us how closely related the query and document are. An angle close to zero means the document and query are closely related. An angle of $90°$ means the query term does not exist in the document.

Let the query term by represent by the vector $\mathbf{q}=(0.25 \quad 0.25 \quad 0.25 \quad 0.25)^T$. Find which of the following documents **d** is the most relevant to this query:

(a) $\mathbf{d}=(1 \quad 1 \quad 1 \quad 3)^T$ (b) $\mathbf{d}=(1 \quad 0 \quad 1 \quad 1)^T$

(c) $\mathbf{d} = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}^T$

Now let $\mathbf{q} = \begin{pmatrix} 0.5 & 0 & 0 & 0.5 \end{pmatrix}^T$. Determine the angle between this vector **q** and the following document vectors:

(d) $\mathbf{d} = \begin{pmatrix} 0 & 2 & 5 & 0 \end{pmatrix}^T$ (e) $\mathbf{d} = \begin{pmatrix} 2 & 0 & 0 & 2 \end{pmatrix}^T$

Comment on your results.

25. In each of the following cases determine whether $B = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ forms a basis for $\mathbb{R}^3$. If *B* is a basis then write the vector **x** in terms of these basis vectors.

(a) $B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} \right\}, \ \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ (b) $B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}, \ \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

26. Let **u** and **v** be vectors in $\mathbb{R}^n$ and matrix $\mathbf{A} = \begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix}$. Write the entries in the matrix $\mathbf{A}^T\mathbf{A}$ in terms of $\|\mathbf{u}\|, \ \|\mathbf{v}\|$ and $\mathbf{u} \cdot \mathbf{v}$. If the vectors **u** and **v** are orthonormal (perpendicular unit) then what is $\mathbf{A}^T\mathbf{A}$ equal to?

27. Let **u** and **v** be non-zero vectors in $\mathbb{R}^n$ such that

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Prove that the vectors **u** and **v** are orthogonal (perpendicular).

28. (a) Let $\mathbf{u} = \begin{pmatrix} 32 & 16 & 8 & 4 & 2 & 1 \end{pmatrix}^T$. Determine the length $\|\mathbf{u}\|$.

(b) Let $\mathbf{u}_n = \begin{pmatrix} r^{n-1} & r^{n-2} & \cdots & r^2 & r & 1 \end{pmatrix}^T$ where $|r| < 1$. Determine the lengths

(i) $\|\mathbf{u}_n\|$ (ii) $\|\mathbf{u}_\infty\|$

[Hint: $1 + r + r^2 + r^3 + \cdots + r^n = \dfrac{1 - r^{n+1}}{1 - r}$ and if $|r| < 1$ then $\lim_{n \to \infty}\left(r^n\right) = 0$.]

29. (a) Give an example of a set of non-unit vectors which form a basis for $\mathbb{R}^4$.

(b) Give an example of 4 distinct vectors which do not form a basis for $\mathbb{R}^4$.

30. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ be a set of linear independent vectors in $\mathbb{R}^n$. Prove that $\{c\mathbf{v}_1, c\mathbf{v}_2, \cdots, c\mathbf{v}_n\}$ where $c \neq 0$ is also linearly independent.

31. Find the first error, if any, in the following derivation:

Let $k_1, k_2, \cdots, k_n$ be scalars and non-zero matrix **A** and vectors **u** be compatible such that

$$k_1(\mathbf{Au}_1)+k_2(\mathbf{Au}_2)+\cdots+k_n(\mathbf{Au}_n)=\mathbf{O}$$
$$\mathbf{A}(k_1\mathbf{u}_1+k_2\mathbf{u}_2+\cdots+k_n\mathbf{u}_n)=\mathbf{O}$$
$$k_1\mathbf{u}_1+k_2\mathbf{u}_2+\cdots+k_n\mathbf{u}_n=\mathbf{O}$$

32. Determine the first error, if any, in the following derivation:

Given that **u** and **v** are vectors in $\mathbb{R}^n$ and $\|\mathbf{u}+\mathbf{v}\|^2=\|\mathbf{u}-\mathbf{v}\|^2$. We have

$$\|\mathbf{u}+\mathbf{v}\|^2=\|\mathbf{u}\|^2+2(\mathbf{u}\cdot\mathbf{v})+\|\mathbf{v}\|^2$$
$$\|\mathbf{u}-\mathbf{v}\|^2=\|\mathbf{u}\|^2-2(\mathbf{u}\cdot\mathbf{v})+\|\mathbf{v}\|^2$$

Equating these gives

$$\|\mathbf{u}\|^2+2(\mathbf{u}\cdot\mathbf{v})+\|\mathbf{v}\|^2=\|\mathbf{u}\|^2-2(\mathbf{u}\cdot\mathbf{v})+\|\mathbf{v}\|^2$$
$$4(\mathbf{u}\cdot\mathbf{v})=0 \quad\Rightarrow\quad \mathbf{u}\cdot\mathbf{v}=0$$

Vectors **u** and **v** are orthogonal.

33. Let **A** be an *n* by *n* matrix and **u**, **v** be vectors in $\mathbb{R}^n$. Prove that

$$\mathbf{Au}\cdot\mathbf{v}=\mathbf{u}\cdot\mathbf{A}^T\mathbf{v}$$

34. Let $\mathbf{u}\in\mathbb{R}^n$ be orthogonal to the following vectors $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_k\}$ in $\mathbb{R}^n$. Prove that **u** is orthogonal to the linear combination

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots+c_k\mathbf{v}_k \text{ where } c\text{'s are scalars}$$

35. Let $\mathbf{u}\in\mathbb{R}^\infty$ be a vector in an infinite dimension Euclidean space such that the sum of the infinite entries (series) converges:

$$u_1^2+u_2^2+u_3^2+\cdots=S \text{ where } S \text{ is a real number}$$

We define the norm (length) as $\|\mathbf{u}\|^2=u_1^2+u_2^2+u_3^2+\cdots$.

(i) Is $\mathbf{v}=\left(\frac{1}{3},\ 0,\ 0,\ 0,\ \cdots\right)$ a member of this infinite Euclidean space $\mathbb{R}^\infty$? Give reasons for your answer.

(ii) Is $\mathbf{u}=\left(1,\ \frac{1}{3},\ \frac{1}{9},\ \frac{1}{27},\ \cdots\right)$ a member of this infinite Euclidean space $\mathbb{R}^\infty$? Give reasons for your answer.

[Hint: $1+x+x^2+x^3+\cdots=\frac{1}{1-x}$ provided $|x|<1$.]

(iii) Determine the length of this vector $\|\mathbf{u}\|$ in part (ii).

(iv) We define $\mathbf{u}\cdot\mathbf{v}=u_1v_1+u_2v_2+u_3v_3+\cdots=T$ where *T* is a real number. Find $\mathbf{u}\cdot\mathbf{v}$ for the given vectors in parts (i) and (ii).

(v) Determine the angle between the vectors **u** and **v**.

Brief Solutions to Problems of Chapter 2

1. No
2. (a) Yes (b) No
4. $\left(\frac{4a-2b+c}{15}\right)\begin{pmatrix}2\\-3\\1\end{pmatrix}+\left(\frac{a+b+c}{3}\right)\begin{pmatrix}0\\1\\2\end{pmatrix}+\left(\frac{7a+4b-2c}{15}\right)\begin{pmatrix}1\\1\\-2\end{pmatrix}=\begin{pmatrix}a\\b\\c\end{pmatrix}$
6. 0.321
8. $\frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix}$
9. $\mathbf{x}=3\mathbf{u}-\mathbf{v}+2\mathbf{w}$
10. $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_4\}$
11. $\frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\end{pmatrix}$
15. (a) Yes (b) No
16. $x_3 \neq 0$ and $x_4 \neq 0$
17. $k \neq 0$ and $k \neq 8$
18. $k=-2,\ -1$. For $k=-2$, $\begin{pmatrix}-7\\2\\-10\end{pmatrix}$ and for $k=-1$, $\begin{pmatrix}-5\\1\\-4\end{pmatrix}$.
19. (a) Independent (b) $\left\{k\mathbf{u}+c\mathbf{v}\ \middle|\ \mathbf{u}=\begin{pmatrix}1\\2\\3\end{pmatrix},\ \mathbf{v}=\begin{pmatrix}1\\0\\3\end{pmatrix}\right\}$ (c) $\mathbf{w}=\begin{pmatrix}1\\0\\0\end{pmatrix}$
20. $\mathbf{w}=\frac{1}{2}\mathbf{u}+\frac{5}{2}\mathbf{v}$
21. $\begin{pmatrix}1/2\\1/8\end{pmatrix}_B$
22. $\left[\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1+\sqrt{3}\end{pmatrix}\right]_B$
23. (a) $\frac{3}{\sqrt{5}}$ (b) $\frac{16}{\sqrt{30}}$
24. Part(c). (d) $90°$ (e) $0°$
25. (a) Not a basis (b) $\mathbf{x}=5\mathbf{b}_1+5\mathbf{b}_2-3\mathbf{b}_3$
26. $\begin{pmatrix}\|\mathbf{u}\|^2 & \mathbf{u}\cdot\mathbf{v}\\ \mathbf{v}\cdot\mathbf{u} & \|\mathbf{v}\|^2\end{pmatrix}$, I

28. (a) 36.946 (b) (i) $\sqrt{\frac{1-r^{2n}}{1-r^2}}$ (ii) $\sqrt{\frac{1}{1-r^2}}$

31. Error occurs in the last line.

32. No Error.

35. (iii) $\frac{3}{\sqrt{8}}$ (iv) $\frac{1}{3}$ (v) $19.47°$